

UNIT

2

FACTORIZATION

- ▶ Factorization
- ▶ Remainder Theorem and Factor Theorem
- ▶ Factorization of Cubic Polynomial

After completion of this unit, the students will be able to:

▶ factorize the expressions of following types.

- Type I: $kx + ky + kz$,
- Type II: $ax + ay + bx + by$,
- Type III: $a^2 \pm 2ab + b^2$,
- Type IV: $a^2 - b^2$,
- Type V: $(a^2 \pm 2ab + b^2) - c^2$,
- Type VI: $a^4 + a^2b^2 + b^4$ or $a^4 + 4b^4$,
- Type VII: $x^2 + px + q$,
- Type VIII: $ax^2 + bx + c$,
- Type IX:
$$\begin{cases} a^3 + 3a^2b + 3ab^2 + b^3, \\ a^3 - 3a^2b + 3ab^2 - b^3, \end{cases}$$
- Type X: $a^3 \pm b^3$,

- ▶ state and apply remainder theorem.
- ▶ find remainder (without dividing) when a polynomial is divided by a linear polynomial.
- ▶ define zeros of a polynomial.
- ▶ state factor theorem and explain through examples.
- ▶ use factor theorem to factorize a cubic polynomial.

2.1 FACTORIZATION OF EXPRESSIONS

Linear Polynomial :-

A polynomial of degree '1' is called a linear polynomial.

For example: $x + 3$, $2x - 5$ etc. The general form of linear polynomials is $ax + b$ where a, b are real numbers and $a \neq 0$.

Quadratic Polynomial :-

A polynomial of degree '2' is called quadratic polynomial e.g.

$3x^2 + 5x - 2$, $4x^2 - 3x + 1$ etc. The general form of a quadratic polynomial is $ax^2 + bx + c$, where a, b, c are real numbers and $a \neq 0$.

Cubic Polynomial :-

A polynomial of degree '3' is called a cubic polynomial. e.g.

$x^3 - 3x^2 + 5x + 2$, $4x^3 + 5x^2 - 2$ etc. The general form of cubic polynomial is $ax^3 + bx^2 + cx + d$ where a, b, c, d are real numbers and $a \neq 0$.

Let $P(x)$ be any polynomial and let a, b, c be any real numbers such that $P(x) = (x-a)(x-b)(x-c)$. Then, clearly each one of $(x-a)$, $(x-b)$, $(x-c)$ is a linear factor of $P(x)$.

To express a given polynomial as the product of linear factors or factors of degree less than that of the given polynomial, is known as factorization.

We see that in $15 = 3 \times 5$, 3 and 5 are factors of 15. Similarly, in $ax + ay = a(x + y)$, a and $(x + y)$ are factors of $ax + ay$ and in $ax + ay + az = a(x + y + z)$, a and $(x + y + z)$ are factors of $ax + ay + az$.

The process of writing an expression as a product of two or more factors is called factorization.

We factorize the expressions of different types.

Following examples will explain the factorization of the expression.

EXAMPLE-1

Factorize the following

(i) $3x + 12y$

(ii) $x^2 + xy$

(iii) $ad + dc + df$

(iv) $2pq + 6p^2q - 4p^3q$

SOLUTION:

(i) $3x + 12y = 3(x + 4y)$

(ii) $x^2 + xy = x(x + y)$

(iii) $ad + dc + df = d(a + c + f)$

(iv) $2pq + 6p^2q - 4p^3q = 2pq(1 + 3p - 2p^2)$

Factorization of the expression of the form:

$$ax + ay + bx + by$$

Following examples will explain the factorization of the expression.

EXAMPLE-2

Factorize the following expressions:

(i) $2ax + bx + 6ay + 3by$

(ii) $2yx + 18y^2 - 3zx + 27zy$

(iii) $5ym + 15yn + 2zm + 6zn$

SOLUTION:

(i) $2ax + bx + 6ay + 3by$

$$= x(2a + b) + 3y(2a + b)$$

$$= (2a + b)(x + 3y)$$

Now check $(2a + b)(x + 3y) = 2ax + bx + 6ay + 3by$

(ii) $2yx + 18y^2 + 3zx + 27zy$

$$= 2y(x + 9y) + 3z(x + 9y)$$

$$= (2y + 3z)(x + 9y)$$

(iii) $5ym + 15yn + 2zm + 6zn$

$$= 5y(m + 3n) + 2z(m + 3n)$$

$$= (5y + 2z)(m + 3n)$$

Factorization of the expression of the form:

$$a^2 \pm 2ab + b^2$$

We know that: (i) $a^2 + 2ab + b^2 = (a + b)^2$

$$(ii) a^2 - 2ab + b^2 = (a - b)^2$$

Expressions which have the pattern of the left hand side of (i) and (ii) are called perfect squares. These identities are useful in helping us to factorize certain expressions. Following examples will explain the factorization of the expressions.

EXAMPLE-3

Factorize the following.

$$(i) x^2 + 6x + 9 \quad (ii) t^2 - 12t + 36$$

SOLUTION: (i) $x^2 + 6x + 9 = x^2 + 2(3)(x) + 3^2$
 $= (x + 3)^2$

$$(ii) t^2 - 12t + 36 = t^2 - 2(6)(t) + 6^2$$
$$= (t - 6)^2$$

Factorization of the expression of the form:

$$a^2 - b^2$$

This is called difference of two squares. $a^2 - b^2 = (a - b)(a + b)$

Following examples will explain the factorization of the expression.

EXAMPLE-4

Factorize the following.

$$(i) k^2 - 81 \quad (ii) 9a^2 - (b + c)^2$$

SOLUTION:

$$(i) k^2 - 81 = k^2 - 9^2$$
$$= (k + 9)(k - 9)$$

$$(ii) 9a^2 - (b + c)^2 = (3a)^2 - (b + c)^2$$
$$= [3a + (b + c)][3a - (b + c)]$$
$$= [3a + b + c][3a - b - c]$$

EXAMPLE-5 Factorize $36d^2 - 1$

SOLUTION: $36d^2 - 1 = (6d)^2 - (1)^2$
 $= (6d + 1)(6d - 1)$

EXERCISE - 2.1**Factorize:**

1- $3a(x + y) - 7b(x + y)$

2- $ax + ay - x^2 - xy$

3- $a^3 + a - 3a^2 - 3$

4- $x^3 + y - xy - x$

5- $3ax + 6ay - 8by - 4bx$

6- $2a^2 - bc - 2ab + ac$

7- $a(a - b + c) - bc$

8- $8 - 4a - 2a^3 + a^4$

9- $16x^2 - 24xa + 9a^2$

10- $1 - 14x + 49x^2$

11- $20x^2 + 5 - 20x$

12- $2a^3b + 2ab^3 - 4a^2b^2$

13- $x^2 + x + \frac{1}{4}$

14- $x^2 + \frac{1}{x^2} - 2$

15- $5x^3 - 30x^2 + 45x$

16- $a^2 + b^2 + 2ab + 2bc + 2ac$

Factorization of the expression of the form:

(i) $(a^2 + 2ab + b^2) - c^2$

(ii) $(a^2 - 2ab + b^2) - c^2$

Following examples will explain the factorization of the expressions.

EXAMPLE-1

Resolve into factors:

$$x^2 + 2xy + y^2 - 4z^2$$

SOLUTION: $(x^2 + 2xy + y^2) - 4z^2$

$$= (x + y)^2 - (2z)^2$$

$$= (x + y - 2z)(x + y + 2z)$$

EXAMPLE-2

Resolve into factors:

$$c^2 + 6bc + 9b^2 - 16x^2$$

SOLUTION:

$$\begin{aligned}
 (c^2 + 6bc + 9b^2) - 16x^2 \\
 &= (c + 3b)^2 - (4x)^2 \\
 &= (c + 3b + 4x)(c + 3b - 4x)
 \end{aligned}$$

EXAMPLE-3

Factorize:

(i) $a^2 - 2ab + b^2 - 9c^2$

(ii) $x^2 - 6xy + 9y^2 - 4z^2$

SOLUTION:

$$\begin{aligned}
 \text{(i)} \quad a^2 - 2ab + b^2 - 9c^2 &= a^2 - 2ab + b^2 - 9c^2 \\
 &= (a - b)^2 - (3c)^2 \\
 &= (a - b - 3c)(a - b + 3c)
 \end{aligned}$$

$$\begin{aligned}
 \text{(ii)} \quad x^2 - 6xy + 9y^2 - 4z^2 &= x^2 - 2(3y)x + 9y^2 - 4z^2 \\
 &= (x - 3y)^2 - (2z)^2 \\
 &= (x - 3y - 2z)(x - 3y + 2z)
 \end{aligned}$$

Factorization of the expression of the form:

(i) $a^4 + a^2b^2 + b^4$

(ii) $a^4 + 4b^4$

EXAMPLE-4

Factorize: $x^4 + x^2 + 1$

$$\begin{aligned}
 \text{SOLUTION:} \quad x^4 + x^2 + 1 &= x^4 + x^2 + 1 + x^2 - x^2 \\
 &= (x^4 + 2x^2 + 1) - x^2 \\
 &= (x^2 + 1)^2 - x^2 \\
 &= (x^2 + x + 1)(x^2 - x + 1)
 \end{aligned}$$

SOLUTION:

$$\begin{aligned}
 x^4 + 64 &= (x^2)^2 + 8^2 + 2(8)x^2 - 2(8)x^2 \\
 &= (x^2 + 8)^2 - 16x^2 \\
 &= (x^2 + 8)^2 - (4x)^2 \\
 &= (x^2 + 8 + 4x)(x^2 + 8 - 4x)
 \end{aligned}$$

EXAMPLE-6

Resolve into factors:

$$x^4 + x^2y^2 + y^4$$

SOLUTION:

$$\begin{aligned}
 x^4 + x^2y^2 + y^4 &= (x^4 + 2x^2y^2 + y^4) - x^2y^2 \\
 &= (x^2 + y^2)^2 - (xy)^2 \\
 &= (x^2 + y^2 + xy)(x^2 + y^2 - xy)
 \end{aligned}$$

EXERCISE - 2.2

Resolve into Factors:

1. $x^2 + 2xy + y^2 - a^2$

2. $4a^2 + 4ab + b^2 - 9c^2$

3. $x^2 + 6ax + 9a^2 - 16b^2$

4. $y^2 - c^2 + 2cx - x^2$

5. $x^2 + y^2 + 2xy - 4x^2y^2$

6. $a^2 - 4ab + 4b^2 - 9a^2c^2$

7. $x^2 - 2xy + y^2 - a^2 + 2ab - b^2$

8. $y^4 + 4$

9. $z^4 + 64y^4$

10. $x^4 + 324$

11. $z^4 - z^2 + 16$

12. $4x^4 - 5x^2y^2 + y^4$

Factorization of the expression of the form:

$$x^2 + px + q$$

Let $x^2 + px + q = (x + r)(x + s)$

Then $x^2 + px + q = x^2 + (r + s)x + rs$

comparing coefficients of the like terms on both sides, we get

$$r + s = p \quad \text{and} \quad rs = q$$

Thus, in order to factorize $x^2 + px + q$, we have to find two numbers 'r' and 's' such that $r + s = p$ and $rs = q$

Following examples will explain the factorization of the expression.

EXAMPLE

Factorize

(i) $x^2 + 7x + 12$

(ii) $x^2 + 4x - 21$

(iii) $x^2 - 5x - 14$

SOLUTION:

(i) In order to factorize $x^2 + 7x + 12$, we must find two numbers 'r' and 's' such that

$$r + s = 7 \quad \text{and} \quad rs = 12$$

Clearly $4 + 3 = 7$ and $4 \times 3 = 12$

$$\therefore x^2 + 7x + 12 = x^2 + 4x + 3x + 12$$

$$= x(x + 4) + 3(x + 4)$$

$$= (x + 4)(x + 3)$$

(ii) In order to factorize $x^2 + 4x - 21$, we must find two numbers 'r' and 's' such that

$$r + s = 4 \quad \text{and} \quad rs = -21$$

Clearly $7 + (-3) = 4$ and $7(-3) = -21$

$$\therefore x^2 + 4x - 21 = x^2 + 7x - 3x - 21$$

$$= x(x + 7) - 3(x + 7)$$

$$= (x + 7)(x - 3)$$

Clearly $-7 + 2 = -5$ and $-7 \times 2 = -14$

$$\begin{aligned}\therefore x^2 - 5x - 14 &= x^2 - 7x + 2x - 14 \\ &= x(x-7) + 2(x-7) \\ &= (x-7)(x+2)\end{aligned}$$

Factorization of the expression of the form:

$$ax^2 + bx + c, a \neq 0$$

To factorize the expression of the form $ax^2 + bx + c$, we find numbers p and q such that $p + q = b$ and $pq = ac$ in the given expression, where a, b, c are constants and $a \neq 0$.

Following examples will explain the factorization of the expression.

EXAMPLE

Factorize: (i) $6x^2 + 7x - 3$ (ii) $\sqrt{3}x^2 + 11x + 6\sqrt{3}$

SOLUTION:

(i) The given expression $6x^2 + 7x - 3$,
is of the form $ax^2 + bx + c$, $ac = 6 \times (-3) = -18$

$$\begin{aligned}\therefore 6x^2 + 7x - 3 &= 6x^2 + 9x - 2x - 3 \\ &= 3x(2x + 3) - 1(2x + 3) \\ &= (2x + 3)(3x - 1)\end{aligned}$$

(ii) $\sqrt{3}x^2 + 11x + 6\sqrt{3}$; $ac = \sqrt{3} \times 6\sqrt{3} = 18$

Clearly $9 + 2 = 11$

$$\begin{aligned}\therefore \sqrt{3}x^2 + 11x + 6\sqrt{3} &= \sqrt{3}x^2 + 9x + 2x + 6\sqrt{3} \\ &= \sqrt{3}x[x + 3\sqrt{3}] + 2[x + 3\sqrt{3}] \\ &= (\sqrt{3}x + 2)(x + 3\sqrt{3})\end{aligned}$$

$$6 \times (-3) = -18$$

Possible Pairs

$$18 \times (-1) = -18$$

$$(-18) \times (1) = -18$$

$$6 \times (-3) = -18$$

$$-6 \times 3 = -18$$

$$-9 \times 2 = -18$$

$$9 \times (-2) = -18$$

Selected Pair

$$9 \times (-2) = -18$$

EXERCISE - 2.3

Factorize:

1. $x^2 + 9x + 20$

2. $x^2 + 5x - 14$

3. $x^2 + 5x - 6$

4. $x^2 - 7x + 12$

5. $x^2 - x - 156$

6. $x^2 - x - 2$

7. $x^2 - 9x - 90$

8. $a^2 - 12a - 85$

9. $98 - 7x - x^2$

10. $y^2 - 11y - 152$

11. $2x^2 + 3x + 1$

12. $3x^2 + 5x + 2$

13. $2x^2 - x - 1$

14. $6x^2 + 7x - 3$

15. $2 - 3x - 2x^2$

16. $8 + 6x - 5x^2$

17. $3u^2 - 10u + 8$

18. $10x^2 - 7x - 12$

19. $5x^2 - 32x + 12$

20. $4\sqrt{3}x^2 + 5x - 2\sqrt{3}$

Factorization of the expression of the form:

$$\left\{ \begin{array}{l} a^3 + 3a^2b + 3ab^2 + b^3 \\ a^3 - 3a^2b + 3ab^2 - b^3 \end{array} \right\}$$

We know that:

(i) $(a + b)^3 = a^3 + 3a^2b + 3ab^2 + b^3$

(ii) $(a - b)^3 = a^3 - 3a^2b + 3ab^2 - b^3$

Following examples will explain the factorization of the expression.

EXAMPLE Factorize: (i) $x^3 + 6x^2 + 12x + 8$ (ii) $x^3 - 6x^2 + 12x - 8$

SOLUTION:

$$\begin{aligned} \text{(i) } x^3 + 6x^2 + 12x + 8 &= (x)^3 + 3(2)(x)^2 + 3(2)^2x + (2)^3 \\ &= (x + 2)^3 \end{aligned}$$

$$\begin{aligned} \text{(ii) } x^3 - 6x^2 + 12x - 8 &= (x)^3 - 3(2)(x)^2 + 3(2)^2x - (2)^3 \\ &= (x - 2)^3 \end{aligned}$$

$$a^3 \pm b^3$$

We know that

$$(i) \quad a^3 + b^3 = (a+b) (a^2 - ab + b^2)$$

$$(ii) \quad a^3 - b^3 = (a-b) (a^2 + ab + b^2)$$

Following examples will explain the factorization of the expression.

EXAMPLE-1

Factorize

$$(i) \quad x^3 + 27 \quad (ii) \quad 8a^3 - 125b^3 \quad (iii) \quad x^6 - y^6 \quad (iv) \quad a^3 - b^3 - a + b$$

SOLUTION:

$$\begin{aligned} (i) \quad x^3 + 27 &= x^3 + 3^3 \\ &= (x+3) (x^2 - 3x + 9) \end{aligned}$$

$$\begin{aligned} (ii) \quad 8a^3 - 125b^3 &= (2a)^3 - (5b)^3 \\ &= (2a - 5b) [(2a)^2 + (2a) \times (5b) + (5b)^2] \\ &= (2a - 5b) [4a^2 + 10ab + 25b^2] \end{aligned}$$

$$\begin{aligned} (iii) \quad x^6 - y^6 &= (x^3)^2 - (y^3)^2 \\ &= (x^3 + y^3) (x^3 - y^3) \\ &= (x+y) (x^2 - xy + y^2) (x-y) (x^2 + xy + y^2) \\ &= (x+y) (x-y) (x^2 - xy + y^2) (x^2 + xy + y^2) \end{aligned}$$

$$\begin{aligned} (iv) \quad a^3 - b^3 - a + b &= (a^3 - b^3) - (a - b) \\ &= (a-b) (a^2 + ab + b^2) - (a-b) \\ &= (a-b) [a^2 + ab + b^2 - 1] \end{aligned}$$

Remember that:

$$(i) \quad a^2 + 2ab + b^2 = (a + b)^2$$

$$(ii) \quad a^2 - 2ab + b^2 = (a - b)^2$$

$$(iii) \quad a^3 + 3a^2b + 3ab^2 + b^3 = (a + b)^3$$

$$(iv) \quad a^3 - 3a^2b + 3ab^2 - b^3 = (a - b)^3$$

$$(v) \quad (x + y)(x^2 - xy + y^2) = x^3 + y^3$$

$$(vi) \quad (x - y)(x^2 + xy + y^2) = x^3 - y^3$$

EXERCISE - 2.4

Factorize:

$$1. \quad 8x^3 - y^3$$

$$2. \quad 27x^3 + 1$$

$$3. \quad 1 - 343x^3$$

$$4. \quad a^3b^3 + 512$$

$$5. \quad 27 - 1000y^3$$

$$6. \quad 27x^3 - 64y^3$$

$$7. \quad x^3y^3 + z^3$$

$$8. \quad 216p^3 - 343$$

$$9. \quad 8x^3 - \frac{1}{27}$$

$$10. \quad a^3 + b^3 + a + b$$

$$11. \quad a - b - a^3 + b^3$$

$$12. \quad x - 8xy^3$$

$$13. \quad x^{12} - y^{12}$$

$$14. \quad 1 - \frac{64p^3}{q^3}$$

$$15. \quad 1 + 64U^3$$

$$16. \quad 8x^3 - 6x - 9y + 27y^3$$

$$17. \quad z^3 + 125$$

$$18. \quad x^9 + y^9$$

$$19. \quad m^6 - n^6$$

$$20. \quad 64x^7 - xa^6$$

$$21. \quad x^3 - 27a^3$$

$$22. \quad x^3 + 27a^3$$

$$P(x) = a_n x^n + a_{n-1} x^{n-1} + a_{n-2} x^{n-2} + \dots + a_1 x + a_0, a_n \neq 0$$

where 'n' is a non-negative integer and the coefficients are constants, is called a polynomial function of degree 'n'.

For example:

(i) $P(x) = a_1 x + a_0$ (is a polynomial function of degree one), $a_1 \neq 0$

(ii) $P(x) = 3x^2 + 5x + 11$ (is a polynomial function of degree two)

(iii) $P(x) = 7x^5 + 2x^4 + 4x^3 + 7x^2 + 5x + 6$ (is a polynomial function of degree 5)

(iv) $P(x) = 5x^5 + \frac{7}{x} + 6 = 5x^5 + 7x^{-1} + 6$ (is not a polynomial function)

EXAMPLE

Divide $P(x) = 2x^4 + 3x^3 - x - 5$ by $x + 2$

SOLUTION:

	$2x^3 - x^2 + 2x - 5$	quotient
divisor	$\rightarrow x + 2$	dividend
	$2x^4 + 3x^3 - x - 5$	
	$\pm 2x^4 \pm 4x^3$	
	$-x^3 - x - 5$	
	$\mp x^3 \qquad \qquad \mp 2x^2$	
	$2x^2 - x - 5$	
	$\pm 2x^2 \pm 4x$	
	$-5x - 5$	
	$\mp 5x \mp 10$	
	5	remainder

Remember that:

$$\text{Dividend} = \text{Divisor} \times \text{quotient} + \text{remainder}$$

2.2.1 The Remainder Theorem

If R is the remainder after dividing the polynomial $P(x)$ by $x - a$, then

$$P(a) = R$$

or

If a polynomial $P(x)$ of degree $n \geq 1$ is divided by polynomial $x - a$, where a is any constant, then remainder is $P(a)$, provided there is no term left containing 'x'.

EXAMPLE-1

If $P(x) = 4x^4 + 10x^3 + 19x + 5$, is divided by $x+3$, then find the remainder.

SOLUTION: $P(x) = 4x^4 + 10x^3 + 19x + 5$

$$x - a = x + 3 \Rightarrow a = -3$$

$$\begin{aligned}\text{Therefore } P(-3) &= 4(-3)^4 + 10(-3)^3 + 19(-3) + 5 \\ &= 4 \times 81 - 10 \times 27 - 57 + 5 \\ &= 324 - 270 - 57 + 5 \\ &= -3 + 5\end{aligned}$$

Thus $P(-3) = 2$

$$\boxed{R = 2}$$

EXAMPLE-2

If $P(x) = 5x^4 + 14x^3 + 3x^2 - 5x - 3$ is divided by $x - 1$, find the remainder.

SOLUTION: $P(x) = 5x^4 + 14x^3 + 3x^2 - 5x - 3$

$$x - a = x - 1 \Rightarrow a = 1$$

$$\begin{aligned}\text{Therefore } P(1) &= 5(1)^4 + 14(1)^3 + 3(1)^2 - 5(1) - 3 \\ &= 5 + 14 + 3 - 5 - 3 \\ &= 14\end{aligned}$$

Thus $P(1) = 14$

$$\boxed{R = 14}$$

2.2.2 Finding Remainder Without Dividing

In the following examples, we learn to find the remainder without division, when a polynomial is divided by a linear polynomial.

EXAMPLE-1

Use the remainder theorem to find the remainder when the first polynomial is divided by the second polynomial.

(i) $x^2 + 3x + 7$, $x + 1$

(ii) $x^3 - 2x^2 + 3x + 3$, $x - 3$

SOLUTION: (i) Let $P(x) = x^2 + 3x + 7$

Since the divisor $= x + 1$

$$\text{Therefore } x - a = x + 1 \Rightarrow a = -1.$$

By the remainder theorem

$$R = P(-1)$$

$$P(-1) = (-1)^2 + 3(-1) + 7$$

$$\text{Now } = 1 - 3 + 7$$

$$R = 5$$

(ii) Let $P(x) = x^3 - 2x^2 + 3x + 3$

$$x - a = x - 3 \Rightarrow a = 3$$

$$R = P(3)$$

$$\text{Now } P(3) = (3)^3 - 2(3)^2 + 3(3) + 3$$

$$= 27 - 18 + 9 + 3$$

$$\boxed{R = 21}$$

EXAMPLE-2

When $x^4 + 2x^3 + kx^2 + 3$ is divided by $x - 2$ the remainder is 1. Find the value of 'k'.

SOLUTION: Let $P(x) = x^4 + 2x^3 + kx^2 + 3$

Since the divisor $= x - 2$, therefore $x - a = x - 2 \Rightarrow a = 2$

Now

$$P(2) = (2)^4 + 2(2)^3 + k(2)^2 + 3$$

$$= 16 + 16 + 4k + 3 = 35 + 4k$$

$$P(2) = 1 \text{ (given)}$$

$$1 = 35 + 4k \Rightarrow 4k = -34 \Rightarrow k = \frac{-17}{2}$$

2.2.3 Zeros of a Polynomial

If $P(x) = x - a_1$ and $Q(x) = x - a_2$ are any first degree polynomials such that $P(a_1) = 0$ and $Q(a_2) = 0$ for polynomials $P(x)$ and $Q(x)$.

Then a_1, a_2 are called zeros of $P(x)$ and $Q(x)$.

2.2.4 The Factor Theorem

If a polynomial $P(x)$ is divided by $x - a$ such that $P(a) = 0$, then $x - a$ is a factor of $P(x)$;

conversely, if $x - a$ is a factor of $P(x)$, then ' a ' is zero of $P(x)$.

EXAMPLE-1

Use the factor theorem to determine if the first polynomial is a factor of the second polynomial.

$$x - 1, \quad x^2 + 4x - 5$$

SOLUTION: Let $P(x) = x^2 + 4x - 5$

$$x - a = x - 1$$

$$\Rightarrow a = 1$$

$$P(1) = 1^2 + 4(1) - 5$$

$$= 1 + 4 - 5$$

$$= 5 - 5$$

$$= 0$$

$$\text{Since } P(1) = 0$$

Thus by factor theorem, $x - 1$ is a factor of $x^2 + 4x - 5$

EXAMPLE-2

Use the factor theorem to show that $x + 1$ is a factor of $P(x) = x^{25} + 1$

SOLUTION: By direct substitution we see that -1 is a zero of $P(x)$

$$P(x) = x^{25} + 1$$

$$\begin{aligned} P(-1) &= (-1)^{25} + 1 \quad \therefore (-1)^{\text{odd}} = -1 \\ &= -1 + 1 \\ &= 0 \end{aligned}$$

Since -1 is a zero of $P(x) = x^{25} + 1$,

The linear polynomial $x - (-1) = x + 1$ is,

by the factor theorem, a factor of $x^{25} + 1$.

EXAMPLE-3

Use the factor theorem to show that $x - 1$ is not a factor of $4x^7 - 2x^6 + x^2 + 2x + 5$

SOLUTION: Let $P(x) = 4x^7 - 2x^6 + x^2 + 2x + 5$

$$x - a = x - 1 \Rightarrow a = 1$$

$$\begin{aligned} P(1) &= 4(1)^7 - 2(1)^6 + 1^2 + 2(1) + 5 \\ &= 4 - 2 + 1 + 2 + 5 \\ &= 10 \neq 0 \end{aligned}$$

Then by factor theorem $x - 1$ is not a factor of $4x^7 - 2x^6 + x^2 + 2x + 5$

EXAMPLE-4

Use the factor theorem to show that $x+1$ is not a factor of $2x^5 - 5x^2 - x + 4$

SOLUTION: Let $P(x) = 2x^5 - 5x^2 - x + 4$

$$x - a = x + 1 \Rightarrow a = -1$$

$$\begin{aligned} P(-1) &= 2(-1)^5 - 5(-1)^2 - (-1) + 4 \\ &= -2 - 5 + 1 + 4 \end{aligned}$$

$$P(-1) = -2 \neq 0$$

$x + 1$ is not a factor of $2x^5 - 5x^2 - x + 4$.

2.3 FACTORIZING A CUBIC POLYNOMIAL

In order to factorize a cubic polynomial, we see the following examples:

EXAMPLE-1

Factorize the following

$$x^3 - x^2 - 10x + 10; \quad x - 1$$

SOLUTION: $P(x) = x^3 - x^2 - 10(x) + 10; \quad x - 1$

$$x - a = x - 1 \Rightarrow a = 1$$

$$P(1) = 1^3 - 1^2 - 10 + 10$$

$$= 0, \text{ therefore } x - 1 \text{ is a factor of } P(x)$$

$$x^2 - 10$$

Now $x - 1 \overline{) x^3 - x^2 - 10x - 10}$

$$\underline{-x^3 + x^2}$$

$$-10x + 10$$

$$\underline{+10x - 10}$$

$$0$$

Now $P(x) = \text{quotient} \times \text{divisor}$

Thus $x^3 - x^2 - 10x + 10 = (x - 1)(x^2 - 10)$

EXAMPLE-2

Factorize the following $x^3 - 8; \quad x - 2$

SOLUTION: $P(x) = x^3 - 8, \quad x - a = x - 2 \Rightarrow a = 2$

$$P(2) = 2^3 - 8 = 8 - 8$$

$$= 0, \text{ therefore } x - 2 \text{ is a factor of } P(x)$$

$$x^2 + 2x + 4$$

Now $x - 2 \overline{) x^3 - 8}$

$$\underline{-x^3 + 2x^2}$$

$$2x^2 - 8$$

$$\underline{-2x^2 + 4x}$$

$$4x - 8$$

$$\underline{-4x + 8}$$

$$0$$

Now $P(x) = x^3 - 8$

$$= (x - 2)(x^2 + 2x + 4)$$

EXERCISE - 2.5

I- Evaluate each of the polynomials for the value indicated.

1. $P(x) = 2x^3 - 5x^2 + 7x - 7$; $P(2)$

2. $P(x) = x^4 - 10x^2 + 25x - 2$; $P(-4)$

3. $P(x) = x^4 + 5x^3 - 13x^2 - 30$; $P(-1)$

4. $P(x) = x^5 - 10x^3 + 7x + 6$; $P(3)$

5. $P(x) = x^4 + 4x^3 - 9x^2 + 19x + 6$; $P(-2)$

II- Determine whether the second polynomial is a factor of the first polynomial without dividing (Hint: evaluate directly and use the factor theorem).

6. $x^{18} - 1$; $x + 1$

7. $x^{18} - 1$; $x - 1$

8. $x^9 - 2^9$; $x + 2$

9. $x^9 + 2^9$; $x - 2$

10. $3x^4 - 2x^3 + 5x - 6$; $x - 1$

11. $5x^6 - 7x^3 - 6x + x$; $x - 1$

12. $3x^3 - 7x^2 - 8x + 2$; $x + 1$

13. $5x^8 - 2x^5 + 3x^3 + 6x + 2$; $x + 1$

14. $6x^3 + 2x^2 - x + 9$; $x - 1$

15. $4x^3 - 3x^2 - 8x + 4$; $x - 2$

16. $5x^3 + 3x^2 - x + 1$; $x + 1$

17. $2y^3 - 8y^2 + y - 4$; $y - 4$

18. $z^3 - 5z^2 - 4z - 4$; $z + 2$

III- Solve.

19. If $P(x) = x^3 - kx^2 + 3x + 5$ is divided by $x - 1$, find k , if remainder is 8.

20. If $P(x) = 3x^3 + kx - 26$ is divided by $x - 2$, find k , if remainder is 0.

Review Exercise-2

I- Encircle the Correct Answer.

1. A linear polynomial is of degree =
(a) 0 (b) 1
(c) 2 (d) 3
2. A quadratic polynomial is of degree =
(a) 0 (b) 1
(c) 2 (d) 3
3. A cubic polynomial is of degree =
(a) 0 (b) 1
(c) 2 (d) 3
4. Factorization of $(x+3)^2 - 4$ is
(a) $(x+1)(x+5)$ (b) $(x-1)(x+5)$
(c) $(x+1)(x-5)$ (d) $(x-1)(x-5)$
5. Factorization of $x^4 - 16$ is
(a) $(x-2)(x+2)$ (b) $(x-4)(x+4)$
(c) $(x-2)(x+2)(x^2+4)$ (d) $(x-2)(x+4)$
6. Factorization of $x^3 - y^3$ is =
(a) $(x-y)(x^2+y^2)$ (b) $(x-y)(x^2+xy+y^2)$
(c) $(x-y)(x^2-xy+y^2)$ (d) $(x+y)(x^2+xy+y^2)$
7. Factorization of $a^4 - 1$ is =
(a) $(a-1)(a+1)(a^2+1)$ (b) $(a-1)(a^2+1)$
(c) $(a+1)(a^2-1)$ (d) $(a^2+1)(a+1)$
8. If a polynomial $P(x)$ of degree $n \geq 1$ is divided by polynomial ' $x-a$ ', where a is any constant, then $P(a)$ is
(a) remainder (b) zero
(c) 1 (d) a

9. If $x - a$ is a factor of $P(x)$, then $P(a) =$

(a) 0

(b) 1

(c) $-a$

(d) a

10. If $P(x) = x^3 - 2x^2 + 5x + 1$, then $P(1) =$

(a) 5

(b) -5

(c) -7

(d) 0

II- Fill in the blanks.

1. A linear polynomial is of degree _____.

2. A quadratic polynomial is of degree _____.

3. A cubic polynomial is of degree _____.

4. Factorization of $x^2 - 9$ is _____.

5. Factorization of $(x+2)^2 - 1$ is _____.

6. Factorization of $x^3 + 8$ is _____.

7. Factorization of $x^3 - 8$ is _____.

8. If $P(x) = x^4 + 3x^2 - 2x + 1$ is divided by $x - 1$, then $P(1) =$ _____.

9. If $P(x) = x^3 + 3x^2 - 3x + 1$ is divided by $x + 2$, then $P(-2) =$ _____.

10. If $P(x) = x^3 - a^3$ is divided by $x - a$, then $P(a) =$ _____.

SUMMARY

Linear Polynomial: A polynomial of degree "1" is called linear polynomial.

Quadratic Polynomial: A polynomial of degree "2" is called quadratic polynomial.

Cubic Polynomial: A polynomial of degree "3" is called cubic polynomial.

Factorization of following types of polynomials:

$$kx + ky + kz, ax + ay + bx + by, a^2 \pm 2ab + b^2$$

$$a^2 - b^2, (a^2 \pm 2ab + b^2) - c^2, a^4 + a^2b^2 + b^4 \text{ or } a^4 + 4b^4,$$

$$x^2 + px + q, ax^2 + bx + c,$$

$$a^3 + 3a^2bx + 3ab^2 + b^3, a^3 - 3a^2b + 3ab^2 - b^3,$$

$$a^3 \pm b^3.$$

Remainder Theorem: If a polynomial $P(x)$ of degree $n \geq 1$ is divided by a polynomial ' $x-a$ ' where ' a ' is any constant, then remainder is $P(a)$.

Factor Theorem: If a polynomial $P(x)$ is divided by ' $x-a$ ' such that $P(a) = 0$, then ' $x-a$ ' is a factor of $P(x)$.