CHAPTER: 2

Scalars and Vectors

2.1 SCALARS

Quantities which can be specified by a number having appropriate units (positive, negative, zero) are called scalars. For example, quantities such as temperature, density, volume. etc are scalars. The number representing any scalars is known as its magnitude. The scalars can be compared only when they have the same physical dimensions (units).

Two or more than two scalars measured in the same system of units are equal only if they have the same magnitude (absolute value) and sign. The scalars are denoted by letters in ordinary type. Operations, with scalars such as, division, subtraction, addition and multiplication follow the rules of elementary algebra.

2.2 VECTORS

Physical quantities having both magnitude and direction with appropriate unit are called vectors. For example, displacement, velocity, acceleration, force, moment of force, electrical field strength, are all vectors, because none of these quantities have a complete meaning without a mention of the direction.

A vector is represented graphically (Fig. 2.1) by a directed line segment or an arrow-head line segment, $\overrightarrow{QP}$, whose length and direction coincide with the magnitude and direction of the quantity under consideration respectively. The tail end-Q is regarded as initial point of the vector and the head-P is called terminal point of the vector.

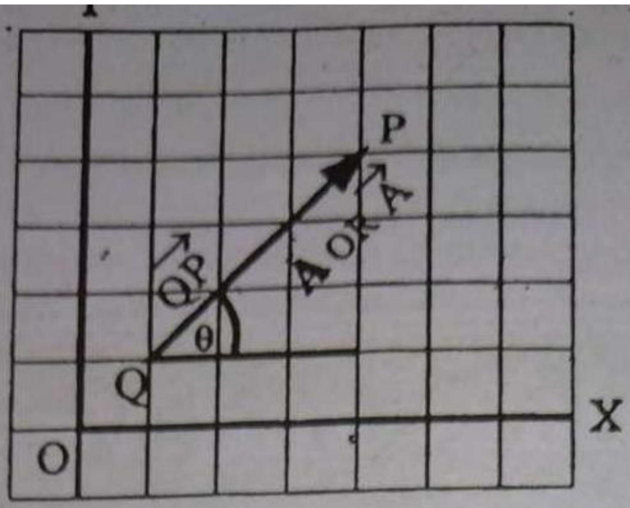


Fig. 2.1 θ - determines direction of the vector $\overrightarrow{QP}$ w.r.t.x-axis

Vectors are denoted by bold faced letters **A**,**B**,**C** and their magnitudes are denoted by |**A**|,|**B**|,|**C**| called the absolute value of **A**,**B**,**C**, respectively; more frequently we represent the magnitude alone by the italic letters, symbol, such as A, B,C respectively. In hand writing, it is convenient to put an arrow above the corresponding letters as $\vec{A}$, $\vec{B}$, $\vec{C}$ and their magnitudes are denoted by A, B,C respectively.

The following definitions are fundamental:

(a) Two vectors $\overrightarrow{OA}$ and $\overrightarrow{OB}$ are equal if they have the same magnitude and similar direction without any consideration of the position of their initial points. fig 2.2(a). Thus

$\overrightarrow{OA} = \overrightarrow{OB}$ when (i) $OA = OB$

(ii) direction of $\overrightarrow{OA}$ is similar to the direction of $\overrightarrow{OB}$.

(b) A vector $-\overrightarrow{OA}$ represents a vector $\overrightarrow{OA}$ with opposite direction i.e the terminal point of vector $\vec{A}$ becomes its initial point and its initial point becomes the terminal point while the magnitude remains same as shown in Fig. 2.2.(b). magnitude

also

$$\overrightarrow{OA} + (-\overrightarrow{OA}) = 0$$

The magnitude of a vector is always treated as non negative and the minus sign indicates the reversal of that vector through an angle of 180°.

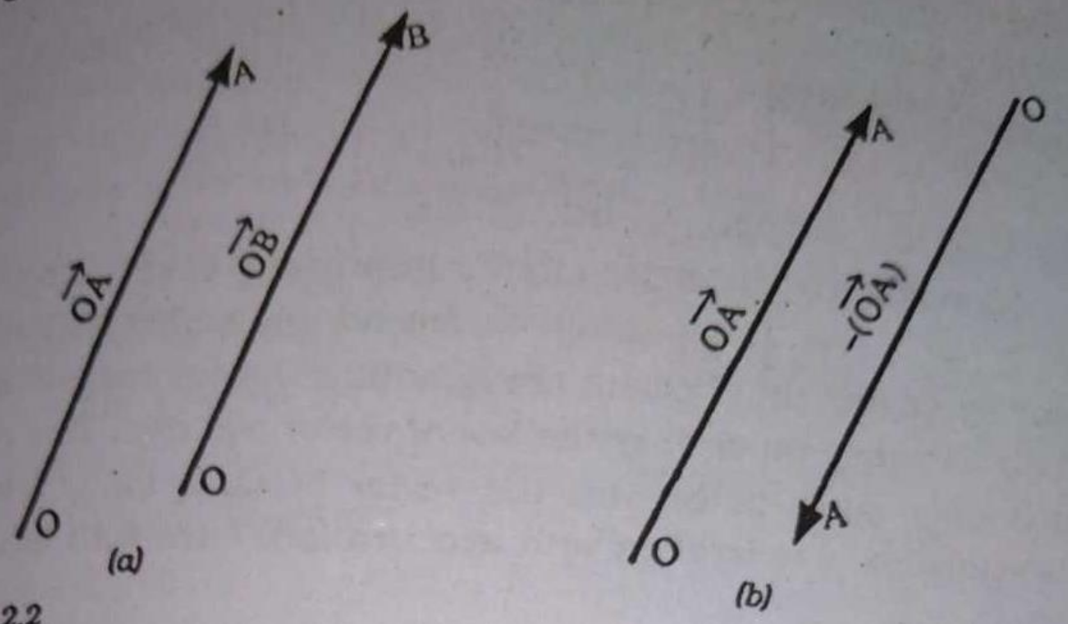


Fig. 2.2

2.3 ADDITION OF VECTORS

Consider two vectors $\overrightarrow{OA}$ and $\overrightarrow{OB}$ starting at a common point O as shown in Fig 2.3. Let these two vectors be the two adjacent sides of a parallelogram, complete the parallelogram OBCA and draw the diagonal OC. Assigning the direction by an arrow head to BC and AC similar to that of $\overrightarrow{OA}$ and $\overrightarrow{OB}$ respectively we get

$$\overrightarrow{BC} = \overrightarrow{OA}$$
$$\overrightarrow{AC} = \overrightarrow{OB}$$

By Definition the sum or resultant of the vectors $\overrightarrow{OB}$ and $\overrightarrow{BC}$ ($\overrightarrow{BC} = \overrightarrow{OA}$) is given by a vector $\overrightarrow{OC}$. (The diagonal of the parallelogram). This is the parallelogram law of vector addition. It is formed by placing the initial point of $\overrightarrow{BC}$ on the terminal point of $\overrightarrow{OB}$ and then joining the terminal point of $\overrightarrow{BC}$ to the initial point of $\overrightarrow{OB}$. The point O is then regarded as the initial point and the point C is regarded as the terminal point of the resultant vector. The direction of the resultant vector, $\overrightarrow{OC}$, is then from the initial point of $\overrightarrow{OB}$ (i.e. the point O) to the terminal point of $\overrightarrow{BC}$ (i.e the point C) as shown in Fig. 2.3 This method of vector addition is known as Head-to-tail rule and can be extended to accomplish addition of any number of vectors. This is also known as triangle law of vector addition.

consequently,

$$\overrightarrow{OC} = \overrightarrow{OA} + \overrightarrow{AC} \qquad 2.1\,(a)$$

and $$\overrightarrow{OC} = \overrightarrow{OB} + \overrightarrow{BC} \qquad 2.1\,(b)$$

Considering the above l.nportant property of vector combination, we are now in a position to amend our earlier definition of vector by saying that vectors are quantities having magnitude, direction and they must obey the law of vector addition. This combination only takes place with the vector of same kind, velocities with velocities, acceleration with acceleration, force with force and so on.

Analytical determination of resultant of two vectors and its direction.

In fig 2.3 consider the triangle OCA. Representing OA, AC and OC by A, B and R respectively, we have by the law of cosines

$$R^2 = A^2 + B^2 - 2AB \cos \angle OAC \qquad 2.2\,(a)$$

$$R = \sqrt{A^2 + B^2 - 2AB \cos \angle OAC} \qquad 2.2\,(b)$$

The Eq. 2.2(b) determines the magnitude, R, of the result vector $\vec{R}$ by the law of sines.

$$\frac{A}{\sin\angle ACO} = \frac{B}{\sin\angle AOC} = \frac{R}{\sin\angle OAC} \qquad 2.3$$

The Eq: 2.3 determines the direction of the resultant vector, $\vec{R}$.

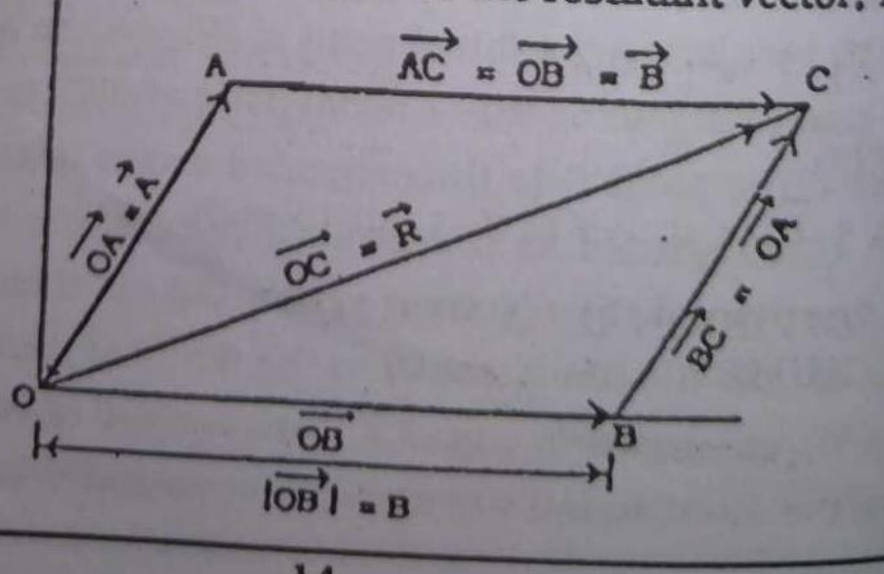


Fig: 2.3

2.4 MULTIPLICATION OF A VECTOR BY A NUMBER

The operation of multiplication of a vector by a number is simple and straight forward. The product of number m, and a vector A as shown in Fig. 2.4 (a), generates a new vector, say $\vec{B}$, whose magnitude is $|m|$ times the magnitude of vector $\vec{A}$, therefore,

$$B = |m| A \qquad 2.4$$

(i) The direction of vector $\vec{B}$, is same as that of vector $\vec{A}$ if m is + ve (Fig. 2.4 [b]).

(ii) The direction of vector $\vec{B}$ is opposite to that of vector $\vec{A}$ if m is - ve (Fig. 2.4 [c]).

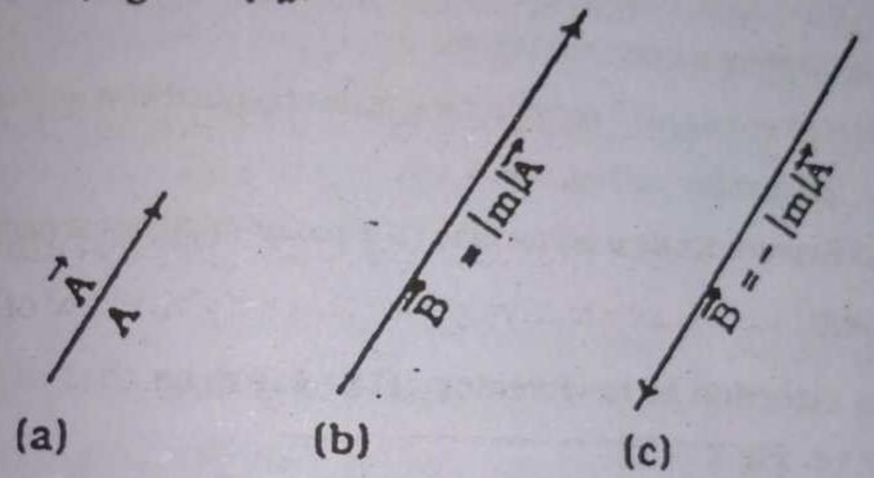


Fig 2.4 *(a) Represents original vector*
(b) Represents new vector after multiplication when scaler mulipier is positive
(c) Represents new vector after multiplication when scaler multiplier, m, is negative

The multiplication of a vector by one or more number (say m,n) is governed by the following rules:

$m\vec{A} = \vec{A}m$; commutative law for multiplication 2.5(a)

$m(n\vec{A}) = (mn)\vec{A}$; associative law for multiplication 2.5(b)

$(m+n)\vec{A} = m\vec{A} + n\vec{A}$; distributive law 2.5(c)

$m(\vec{A} + \vec{B}) = m\vec{A} + m\vec{B}$; distributive law 2.5(d)

2.5 DIVISION OF A VECTOR BY A NUMBER (Non zero)

The division of a vector $\vec{A}$, by a number, n, is simple and involves the multiplication of the vector by the reciprocal of the number n, with the result a new vector is generated. Let n represents a number and its reciprocal $m = {}^{1}/_{n}$, then the magnitude of new vector (say $\vec{B}$) is given by

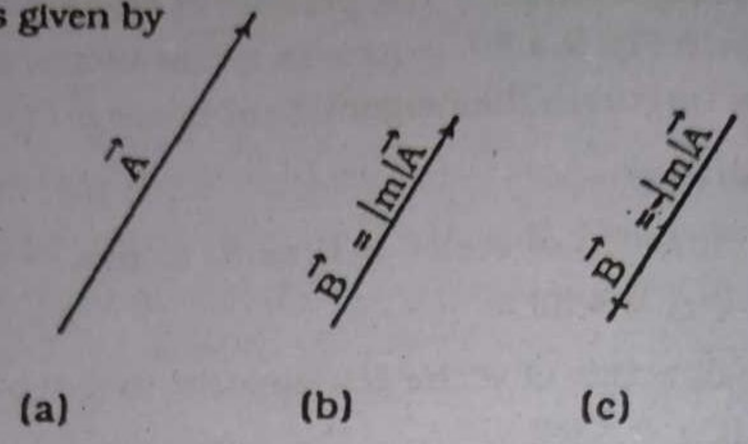


Fig: 2.5 (a) Represents original Vector

(b) Represents new vector after division by number n = 1/m or by number multiplier m = 1/n

(C) Represents new vector when the scalar multiplier is negative.

$\vec{B} = |m|A$

(i) The direction of new vector $\vec{B}$ is same as that of $\vec{A}$, if the number is + ve. Fig 2.5 (b)

(ii) The direction of new vector $\vec{B}$ is opposite to that of $\vec{A}$, if the multiplier is - ve, Fig. 2.5 (c).

2.6 UNIT VECTOR

A vector $\vec{A}$ in any given direction and whose magnitude is unity (A = 1) is referred as a unit vector. We use special notation $\hat{a}$

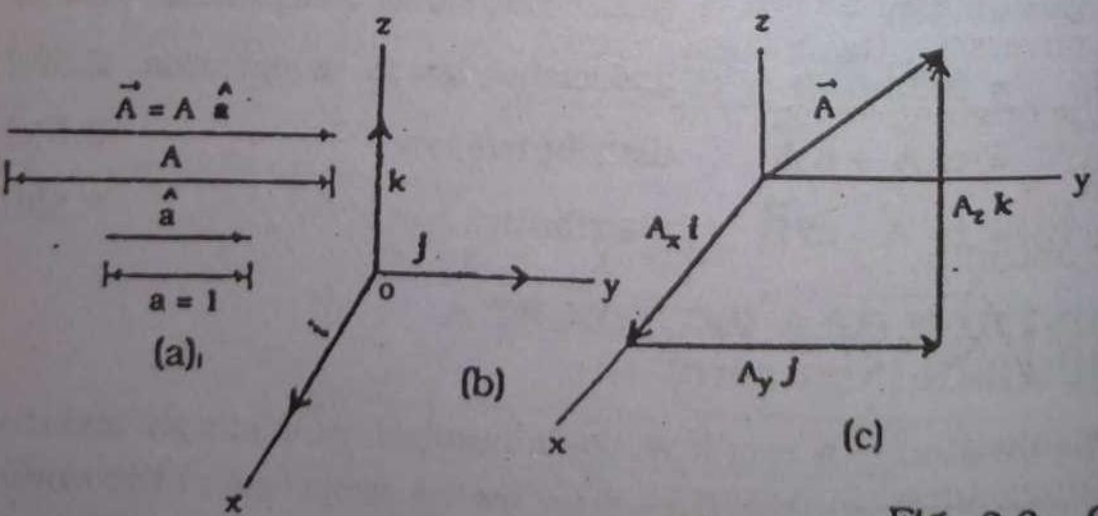


Fig. 2.6 a, b, c

(read as 'a hat') to represent the unit vector. A unit vector can be obtained by dividing the vector by its magnitude i.e.

$$\frac{\vec{A}}{A} = \hat{a} \qquad 2.6\ (a)$$

unit vector only specifies the direction of a given vector.

also $\vec{A} = A\ \hat{a}$ 2.6(b)

The vector $\vec{A}$ which has magnitude A, is just A times the unit vector $\hat{a}$ and has the same direction as $\hat{a}$ as shown in Fig:2.6 (a).

An important set of unit vectors are those having the directions of the positive x, y, and z axes of a three dimensional rectangular coordinate system, and are denoted by i, j and k respectively as shown in figure 2.6 (b). These are called rectangular unit vectors.

Let A_x, A_y and A_z be the rectangular coordinates of the terminal point of a vector $\vec{A}$ with its initial point placed at the origin of a rectangular coordinate system as shown in Fig. 2.6 (c). Then by definition [Eq 2.6(b)] the vectors $|\vec{A}_x|i$, $|\vec{A}_y|j$ and $|\vec{A}_z|k$ are referred as the rectangular component vectors of the vector $\vec{A}$ in the direction of positive x.y and z axes respectively. Also $|\vec{Ax}|$, $|\vec{Ay}|$ and $|\vec{Az}|$ are called rectangular components of $\vec{A}$ along positive x.y and z axes respectively.

Conversely, the sum of rectangular components vectors produces the original vector $\vec{A}$, i.e

$$\vec{A} = A_x i + A_y j + A_z k \qquad 2.7$$

and the magnitude of $\vec{A}$ is given by

$$A = \sqrt{A_x^2 + A_y^2 + A_z^2} \qquad 2.8$$

Here we take $i^2 = j^2 = k^2 = 1$ This shall be explain when we deal with the multiplication of 0 vector by a vector.

Example 2.1

Find the unit vector parallel to the vector.

$$\vec{A} = 3i + 6j - 2k$$

Solution

Using Eq. 2.8 the magnitude of vector $\vec{A}$ is given by

$$|\vec{A}| = \sqrt{(3)^2 + (6)^2 + (-2)^2} = 7$$

then the unit vector parallel to $\vec{A}$ is given by Eq.2.6(a)

$$\hat{a} = \frac{\vec{A}}{|\vec{A}|} = \frac{3i + 6j - 2k}{7}$$

$$\hat{a} = \frac{3}{7}i + \frac{6}{7}j - \frac{2}{7}k$$

By definition the magnitude of unit vector is 1 and therefore we can check our result by evaluating the magnitude of unit vector, i.e

$$\left|\frac{3}{7}i + \frac{6}{7}j - \frac{2}{7}k\right| = \sqrt{\left(\frac{3}{7}\right)^2 + \left(\frac{6}{7}\right)^2 + \left(\frac{-2}{7}\right)^2} = 1$$

2.7 FREE VECTOR

A vector such as the velocity of a body undergoing uniform translational motion, which can be displaced parallel to itself and applied at any point, is known as a FREE VECTOR Fig.2.7. It can be specified by giving its magnitude and any two of the angles between the vector and the coordinate axes. In three dimensions a free vector is uniquely determined by its three projections on the axes of a rectangular coordinate system.

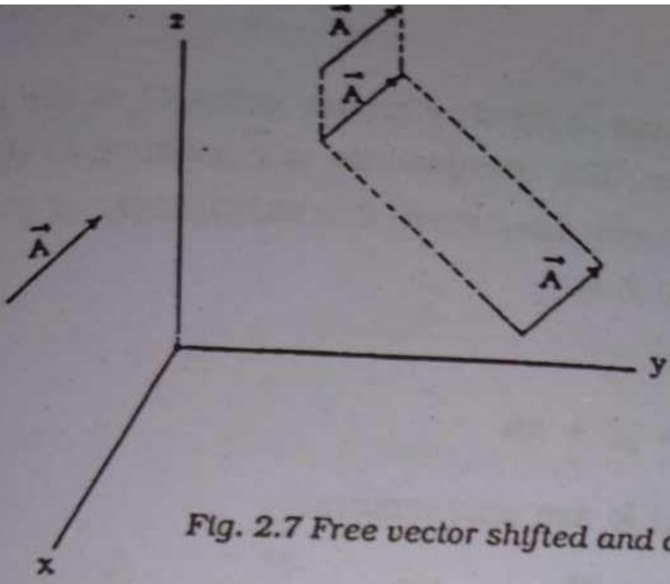


Fig. 2.7 Free vector shifted and drawn parallel to itself

2.8 POSITION VECTOR

Suppose we have a fixed reference point O, then we can specify the position of a given point P w.r. to the point O by means of a vector having magnitude and direction represented by a directed line segment $\overrightarrow{OP}$ as shown in Fig. 2.8(a). This vector is called position vector. We call $\overrightarrow{OP}$ a position vector, since it determines the position of the point P relative to the fixed point O.

Let $\vec{r}$ be a position vector of a point P relative to a rectangular coordinate system defined by unit vectors $\boldsymbol{i}, \boldsymbol{j}, \boldsymbol{k}$ and starting at

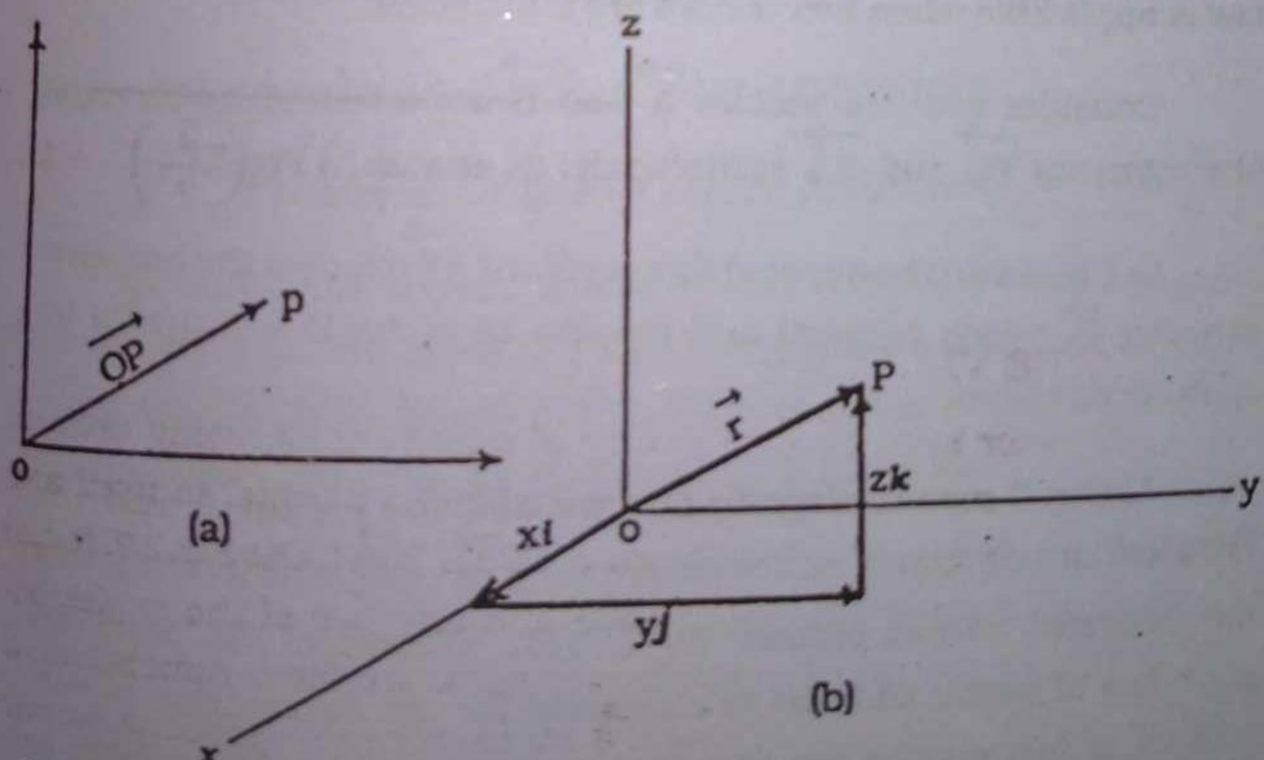


Fig. 2.8 (a) A position vector.
(b) A position vector in rectangular coordinate system.

the origin (corresponds to fixed reference point O) of the rectangular coordinate system. The components of $\vec{r}$ relative to the rectangular coordinate system are called COORDINATES of P and are usually denoted by x. y. z.

Thus

$$\vec{r} = x\hat{i} + y\hat{j} + z\hat{k} \qquad 2.9$$

Where $\hat{i}$, $\hat{j}$ and $\hat{k}$ are unit vectors.

The magnitude of $\vec{r}$ is given by

$$r = |\vec{r}| = \sqrt{x^2 + y^2 + z^2} \qquad 2.10$$

It is important to note that while all vectors have components, only the components of the position vectors are known as coordinates.

2.9 NULL VECTOR

We have seen that the vectors combine or add according to the parallelogram law. We would like to examine whether the same law is applicable when two vectors are subtracted.

Consider two free vectors $\vec{A}$ and $\vec{B}$ represented by directed line segments $\vec{PQ}$ and $\vec{RS}$ respectively, as shown in Fig.2.9(a).

In Fig. 2.9(b) the directed line segment $\vec{XY}$ denotes the negative of vector $\vec{B}$, which is equal and parallel to vector $\vec{B}$ but drawn in opposite direction.

Using the parallelogram law we add the vectors $\vec{A}$ and $-\vec{B}$. The resultant or sum of vectors is given by $\vec{A} + (-\vec{B})$ which represents the difference of two vectors Fig 2.9 (c). Therefore, the parallelogram law of vector addition is also valid for the subtraction of vectors i.e, if two vectors are identical in magnitude and opposite in direction, then difference vector $\vec{A} + (-\vec{B})$ is called NULL or ZERO vector. The null vector has zero magnitude and has no direction or it may have any direction.Nevertheless we shall accept it as vector though it really does not quite fit to our definition of a vector.

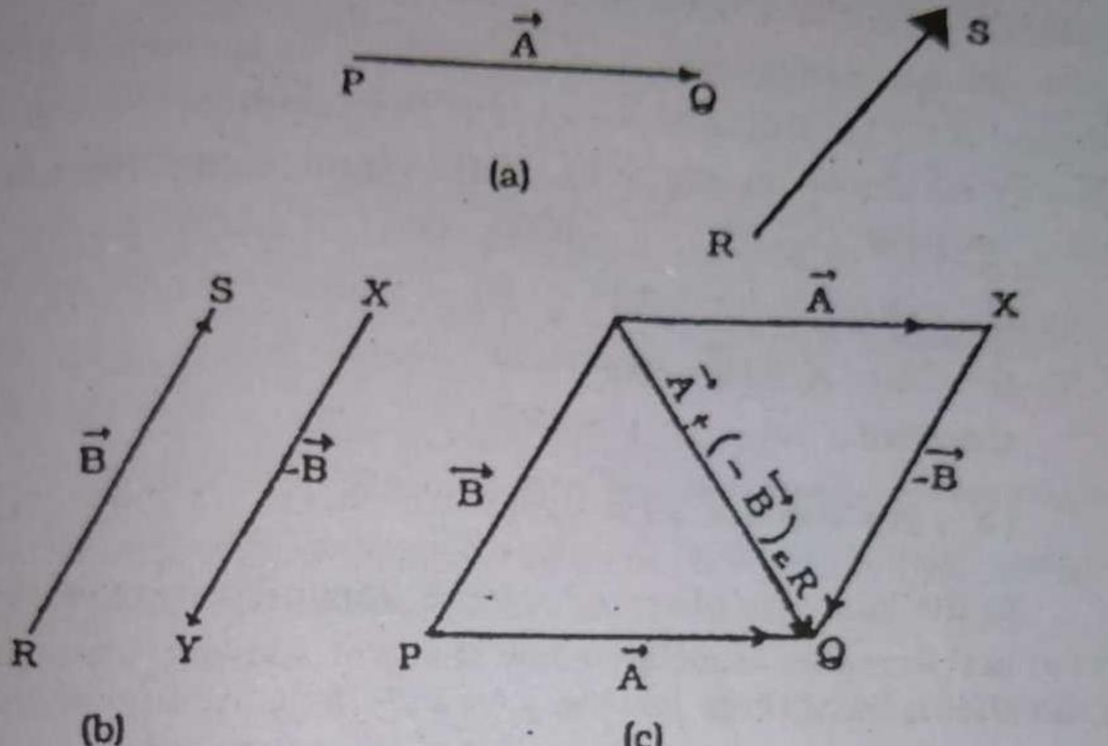


Fig.2.9 (a) Two free Vectors $\vec{A}$ and $\vec{B}$
(b) Negative of Vector $-\vec{B}$
(c) Difference Vector $\vec{A} + (-\vec{B})$

2.10 PROPERTIES OF VECTOR ADDITION

(i) *Commutative law of vector addition.*

Consider two vectors $\vec{A}$ and $\vec{B}$. Let these two vectors represent the two adjacent sides of a parallelogram. We construct the parallelogram OACB as shown in Fig. 2.10, then the diagonal OC represents the resultant vector $\vec{R}$. From Fig. 2.10 we have

$$\vec{R} = \vec{A} + \vec{B} \qquad 2.11\,(a)$$

$$\vec{R} = \vec{B} + \vec{A} \qquad 2.11\,(b)$$

therefore

$$\vec{A} + \vec{B} = \vec{B} + \vec{A} \qquad 2.12$$

In the language of vector algebra, this fact is referred as the commutative law of vector addition.

(ii) Associative law of vector addition.

Consider three vectors $\vec{A}$, $\vec{B}$ and $\vec{C}$, as shown in Fig. 2.11. using HEAD-TO-TAIL rule, we obtain the resultant $(\vec{A}+\vec{B})$ and $(\vec{B}+\vec{C})$ as shown in Fig 2.11. Once again using Head-to-tail rule, we write

$$\vec{R} = (\vec{A}+\vec{B})+\vec{C} \qquad 2.13$$

$$\vec{R} = \vec{A}+(\vec{B}+\vec{C}) \qquad 2.14$$

therefore

$$(\vec{A}+\vec{B})+\vec{C} = \vec{A}+(\vec{B}+\vec{C}) \qquad 2.15$$

In the language of vector algebra, this property of vector addition is referred as associative law of vector addition. Consequently, on the basis of these laws we conclude that the sum of vectors remains same irrespective of any order or grouping of vectors.

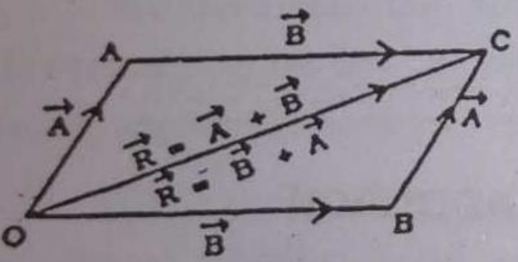


Fig. 2.10
Commutative law of vector addition

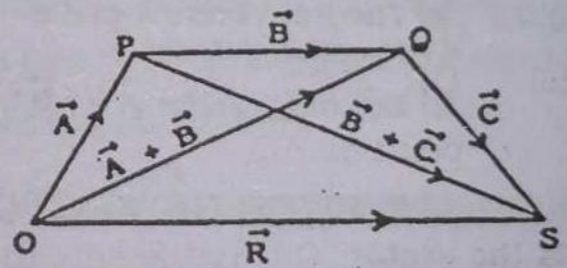


Fig 2.11
Associative law of vector addition

Example 2.2

Represent graphically three displacement vectors $\overrightarrow{OA}$, $\overrightarrow{OB}$, $\overrightarrow{OC}$ having magnitudes OA, OB, OC and making angles of 0°, 40°, 70° with respect to positive x-axis (measured counter clockwise) respectively. Determine the resultant displacement vector and its direction w.r.t. x- axis

Solution

Choosing the appropriate scale of magnitude, the required vectors are drawn as shown in Fig. 2.12(a). Fig. 2.12(b) and Fig 2.12(c). For the determination of the resultant displacement vector we proceed as follows:

Step 1. First form the resultant displacement vector $\overrightarrow{OQ}$ by combining $\overrightarrow{OA}$ and $\overrightarrow{OB}$, according to the parollelogram law as shown in Fig 2.12(d). The magnitude and direction of $\overrightarrow{OQ}$ can easily be measured.

Step 2. Combine the vector $\overrightarrow{OQ}$ and $\overrightarrow{OC}$ by the law of parallelogram and obtain the resultant displacement vector $\overrightarrow{OR}$ as shown in Fig. 2.12(c). The magnitude and direction of $\overrightarrow{OR}$ can easily be measured.

Alternatively, we can obtain the resultant displacement vector $\overrightarrow{OR}$ by assuming all vectors as free vectors and simply using the Head-to-tail rule as shown in Fig. 2.12 (f) observe that

(i) $\overrightarrow{AQ} = \overrightarrow{OB}$, whose initial point lies on the terminal point of $\overrightarrow{OA}$

(ii) $\overrightarrow{QR} = \overrightarrow{OC}$, whose initial point lies on the terminal point of $\overrightarrow{AQ}$

As the vector $\overrightarrow{OR}$ represents the resultant vector, its magnitude $|\overrightarrow{OR}|$ is given by its length and the direction is given by $\angle ROA$.

From Fig.2.12(f). we observe

$$\overrightarrow{OR} = \overrightarrow{OA} + \overrightarrow{OB} + \overrightarrow{OC}$$

$$OR \neq OA + OB + OC$$

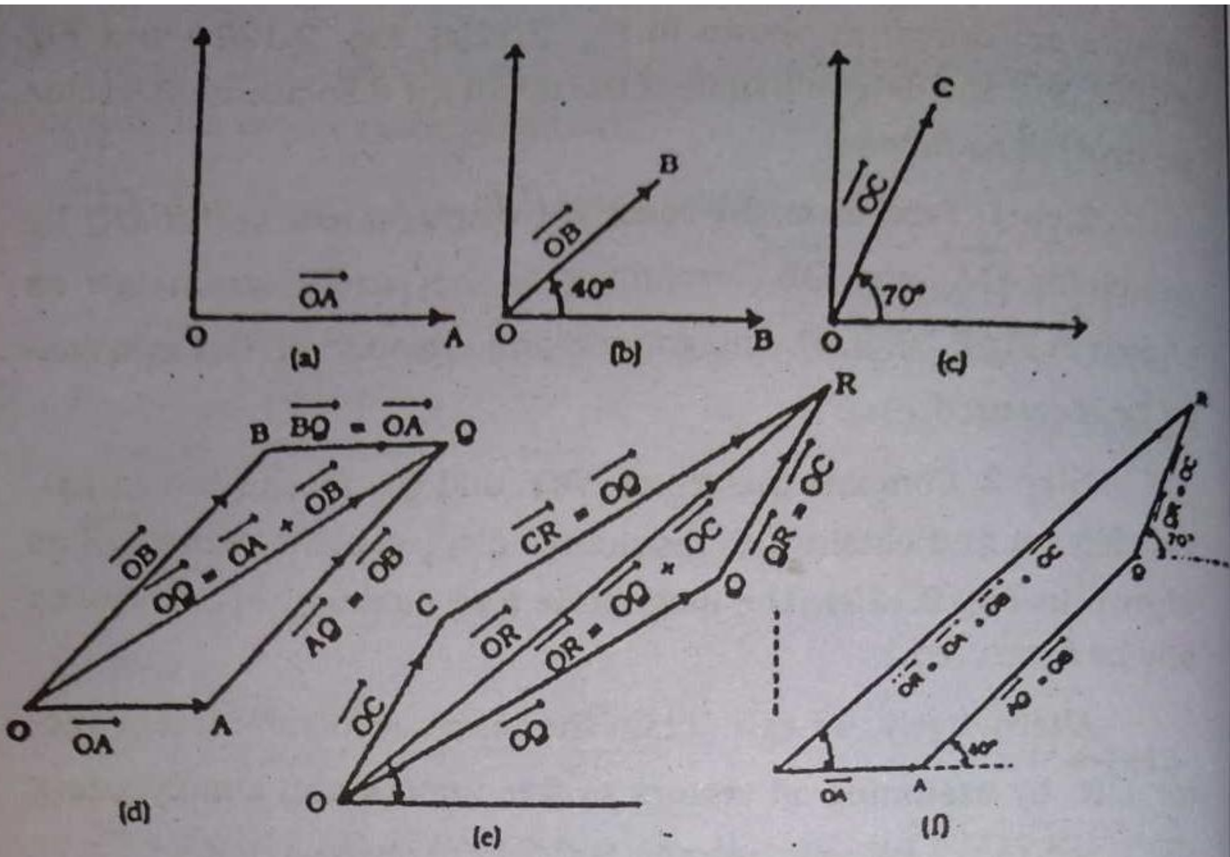


Fig. 2.12

(a) A directed line segment $\overrightarrow{OA}$

(b) A directed line segment $\overrightarrow{OB}$

(c) A directed line segment $\overrightarrow{OC}$

(d) Sum of vectors $\overrightarrow{OA}$ and $\overrightarrow{OB}$

(e) Sum of vectors $\overrightarrow{OA}$, $\overrightarrow{OB}$ and $\overrightarrow{OC}$

(f) Sum of vectors $\overrightarrow{OA}$, $\overrightarrow{OB}$ and $\overrightarrow{OC}$ obtained by applying Head-to-tail rule.

Example 2.3

A point P is subjected to six different forces such as $\overrightarrow{F_1}$, $\overrightarrow{F_2}$, $\overrightarrow{F_3}$, $\overrightarrow{F_4}$, $\overrightarrow{F_5}$, $\overrightarrow{F_6}$ as shown in Fig. 2.13 (a). Find out the resultant force at the point P.

Solution

Assuming all vectors as free vectors, we begin by first drawing the vector $\overrightarrow{F_1}$ parallel to itself as shown in Fig 2.13 (b) Applying Head-to-tail rule, we place the initial point of vector $\overrightarrow{F_2}$ on the terminal point of $\overrightarrow{F_1}$ in such a way that the vector $\overrightarrow{F_2}$ remains parallel to itself. Then we place the initial point of vector $\overrightarrow{F_3}$ on the terminal point of vector $\overrightarrow{F_2}$ while maintaining the original direction and magnitude of the Vector $\overrightarrow{F_3}$ and so on, till the initial point of $\overrightarrow{F_6}$ is placed on the terminal point of $\overrightarrow{F_5}$. Finally, join the terminal point of vector $\overrightarrow{F_6}$ with the point P and this gives the resultant vector R which is equal to

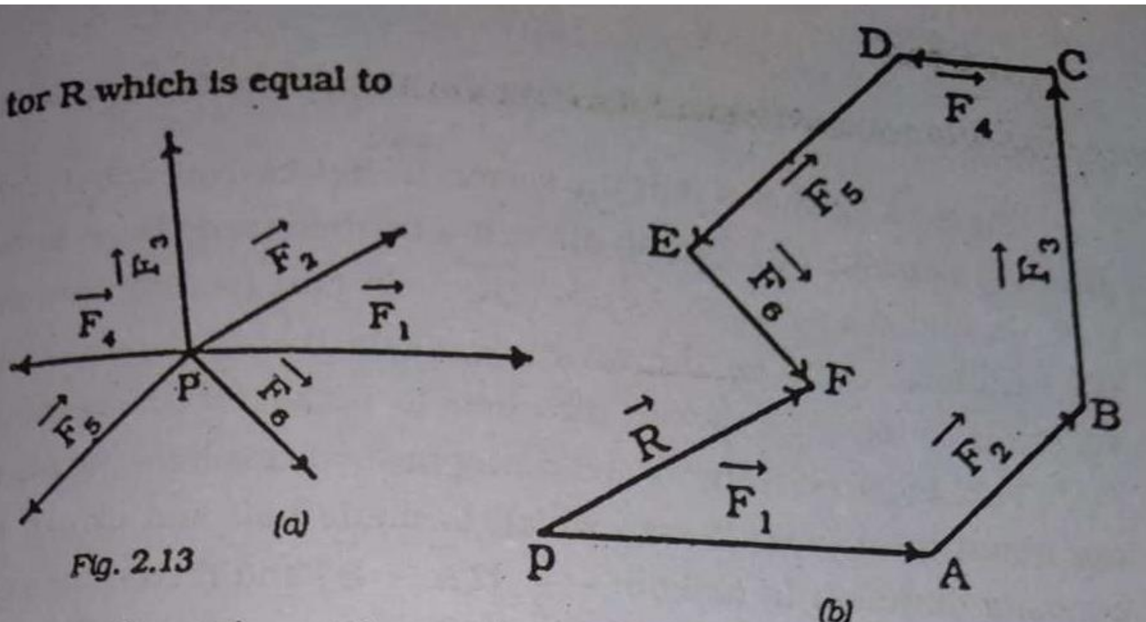


Fig. 2.13

$$\vec{R} = \vec{F}_1 + \vec{F}_2 + \vec{F}_3 + \vec{F}_4 + \vec{F}_5 + \vec{F}_6$$

The force equal and opposite to $\vec{R}$ (i.e $-\vec{R}$) when applied on the point P will prevent any displacement of the point P.

Example 2.4

Given three vectors $\vec{A}$, $\vec{B}$ and $\vec{C}$ as shown in Fig.2.14(a), construct

(a) $\vec{A} - 2\vec{B} + 2\vec{C}$ (b) $4\vec{C} - \frac{1}{2}(2\vec{A} - \vec{B})$

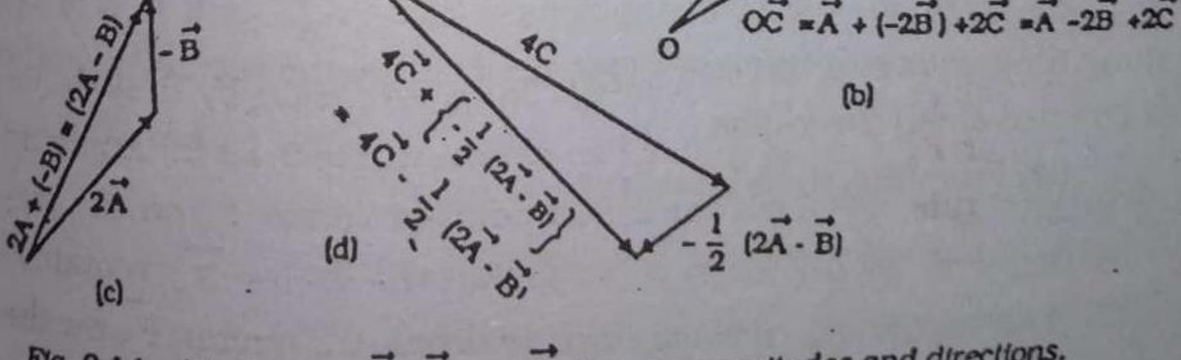


Fig. 2.14 (a) Vectors $\vec{A}$. $\vec{B}$ and $\vec{C}$ of given magnitudes and directions.

(b) Graphical construction to obtain $\vec{A} - 2\vec{B} + 2\vec{C}$.

(c) Graphical construction to obtain $2\vec{A} - \vec{B}$.

(d) Graphical construction to obtain $4\vec{C} - \frac{1}{2}(2\vec{A} - \vec{B})$

Solution

Fig.2.14 (b) shows that the vector $\vec{B}$ has become double and drawn in opposite direction to make it $-2\vec{B}$, then using Head-to-tail rule, we find the resultant vector $\overrightarrow{OC}$. In part (b), first we form the resultant vector of the quantity inside the parentheses in which $\vec{B}$ is drawn opposite direction to make $-\vec{B}$ as shown in Fig.2.14(c).Fig.2.14(d) shows the newly formed resultant vector of the quantity in Parentheses which is made half and drawn in opposite direction to make it $-\frac{1}{2}(2\vec{A} - \vec{B})$ and then combined with $4\vec{C}$ to form the resultant vector.

2.11 RESOLUTION AND COMPOSITION BY RECTANGULAR COMPONENTS

The graphical method already discussed for the addition of vectors is inconvenient for vectors defined in two or three dimensions. Keeping in view this situation we now discuss a method for addition of vector, which is analytical. Here the given vector is resolved into components w.r.to a particular coordinate system.

Consider a vector $\vec{A}$ whose initial point is placed at the origin of two dimensional coordinate system. Fig.2.15(a).

(i) From the terminal point P of the vector $\vec{A}$ fig.2.15(b), we draw two perpendicular lines PQ and PS on x-axis and y-axis, respectively .

(ii) The line OQ is denoted by vector . $\vec{A}_x$, as it is directed along the x-axis and the line PQ is denoted by the vector, $\vec{A}_y$, and it is directed along the y-axis.

(iii) From Fig. 2.15(b) we see .

$$\vec{A} = \vec{A}_x + \vec{A}_y$$

The vectors $\vec{A}_x$ and $\vec{A}_y$ are referred, as rectangular vector components.

From the fig.2.15 (c)

$$A_x = A\cos\theta \qquad 2.16\ (a)$$

$$A_y = A\sin\theta \qquad 2.16\ (b)$$

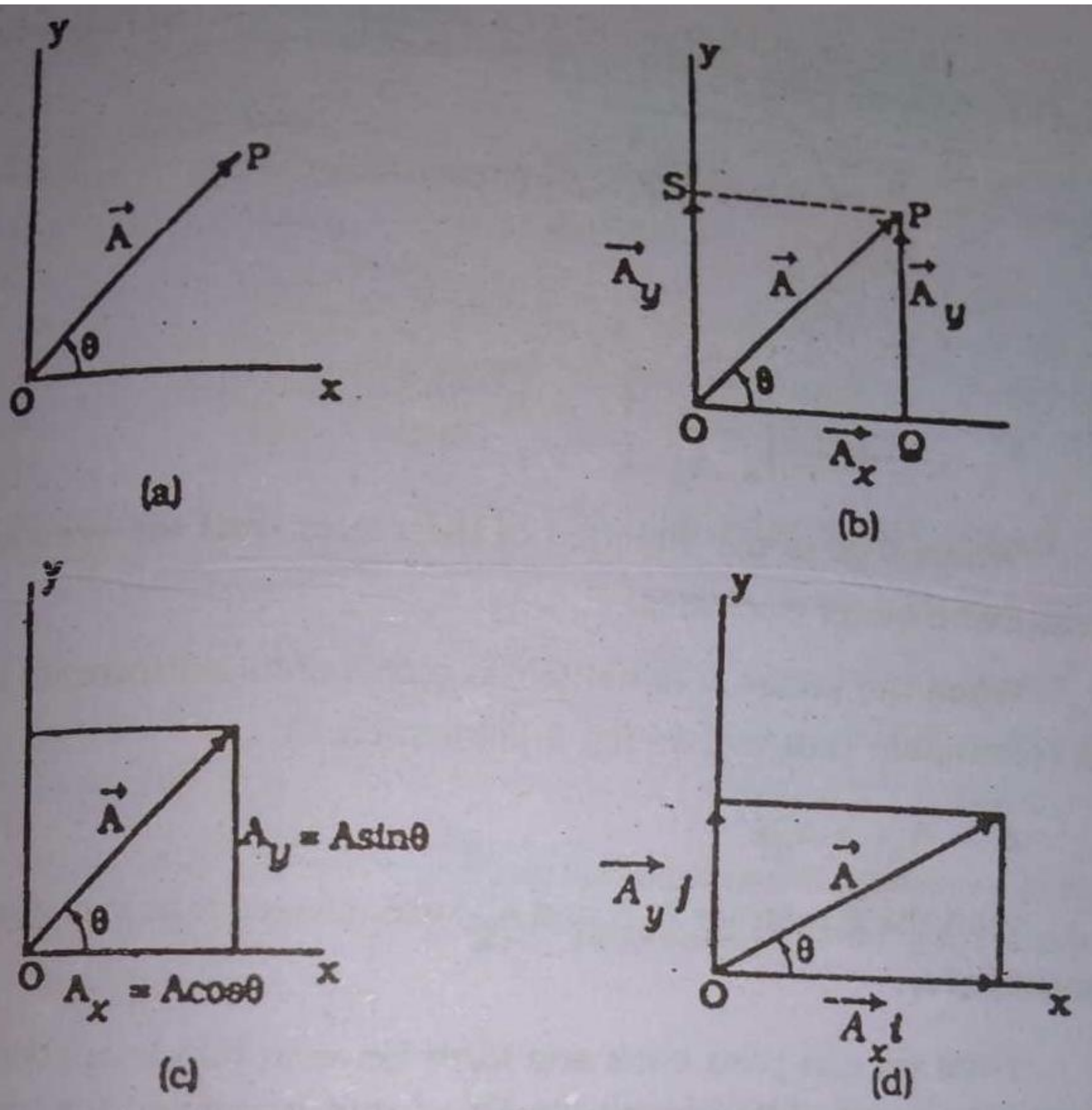


Fig. 2.15 (a) A vector $\vec{A}$ and perpendicular x and y axes
(b) The components of $\vec{A}$ are $\vec{A}_x$ and $\vec{A}_y$
(c) Resolution of vector $\vec{A}$ into its scalar components
(d) Resolution of vector $\vec{A}$ into its vector components

where A, A_x and A_y represent the magnitudes of $\vec{A}$, $\vec{A}_x$ and $\vec{A}_y$ respectively.

Once a vector is resolved into its rectangular components, the components are then used to specify the vector. These components of a vector behave like scalar quantities as depicited in Fig. 2.15 (c)

Conversely, we can obtain the orginal vector once its components are known. That is, we can obtain the magnitude of the vector and its direction from the knowledge of its components. The process by which a vector can be reconstituted from its components is know as composition of a vector. To obtain magnitude and dirction, we refer to Fig. 2.15 (b)

$$A = \sqrt{A_x^2 + A_y^2} \qquad 2.17$$

$$\tan\theta = \frac{A_y}{A_x} \qquad 2.18$$

$$\theta = \tan^{-1}\left[\frac{A_y}{A_x}\right] \qquad 2.19$$

Where θ gives the direction of the vector w.r.t the +ve x-axis measured counter clockwise.

When the vector $\vec{A}$ is written in terms of its components and the rectangular unit vectors fig. 2.15(d) such as

$$\vec{A} = A_x\hat{i} + A_y\hat{j} \qquad 2.20$$

then the quantities $A_x\hat{i}$ and $A_y\hat{j}$ are referred to as vector components of $\vec{A}$.

Thus we can pass back and forth between the description of a vector in terms of its magnitude A and direction θ and the quivalent description in terms of its components.

Having dealt with the resolution and the composition of a vector, we now turn toward its application. The resolution and composition of a vector provide an analytical tool for addition of any number of vectors in a given coordinate system. Once again we restrict our discussion to two-dimensional coordinate system. The method follows as under:

Step: 1 Resolve each given vector into its rectangular components i.e, x-component and y-component.

Step 2. Find the algebraic sum of all the individual x-components, the sum then represents the component of the sum vector along x-axis.

Step: 3 Find the algebraic sum of all the individual y-components, the sum then represents the component of the sum vector along y-axis.

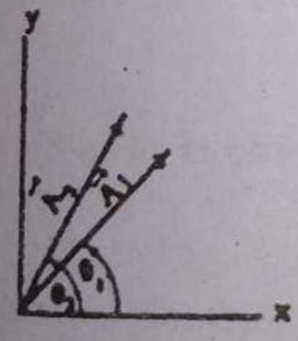
Step: 4 Find the magnitude of the sum vector or of the resultant vector by Eq.No.2.17

Step: 5. Find the direction (i.e the value of angle θ w.r.t. +ve x-axis measured counter clockwise of the resultant vector.

2.12 ADDITION OF VECTORS BY RECTANGULAR COMPONENTS

Consider two vectors $\vec{A}_1$ and $\vec{A}_2$, having magnitude A_1 and A_2 respectively. The vector $\vec{A}_1$ makes an angle θ_1 and the vector $\vec{A}_2$ makes an angle θ_2 with the + ve x-axis as shown in Fig 2.16 (a).

(i) Resolve the vector $\vec{A}_1$ into its rectangular components $\vec{A}_{1x}$ and $\vec{A}_{1y}$ as shown in Fig. 2.16(b). The magnitude of these component vectors is given by



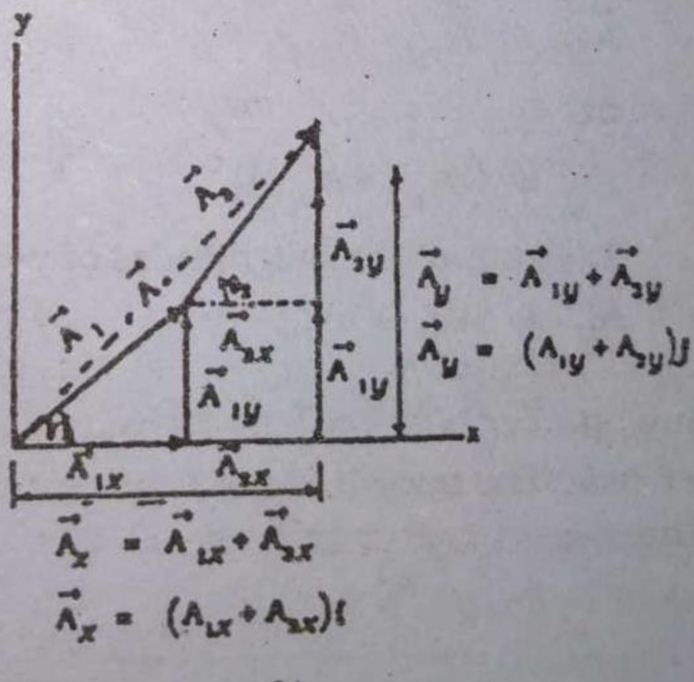


Fig 2.16 (a) Vectors $\vec{A}_1$ and $\vec{A}_2$ and perpendicular x and y axes.

(b) Resolution of vectors $\vec{A}_1$ and $\vec{A}_2$ into their components.

$$A_{ix} = A_i\cos\theta_1 \qquad 2.21\,(a)$$

$$A_{iy} = A_i\sin\theta_1 \qquad 2.21\,(b)$$

(ii) Move the vector $\vec{A}_2$ parallel to itself, so that its initial point lies on the terminal point of vector $\vec{A}_1$ as shown in Fig. 2.16(b).

(iii) Resolve the vector $\vec{A}_2$ into its rectangular components $\vec{A}_{2x}$ and $\vec{A}_{2y}$ as shown in Fig. 2.16(b) then magnitude of each component is given by

$$A_{2x} = A_2\cos\theta_2 \qquad 2.22\,(a)$$

$$A_{2y} = A_2\sin\theta_2 \qquad 2.22\,(b)$$

(iv) The resultant vector along x-axis is given by the algebraic sum of the component vectors along x-axis

$$\vec{A}_x = (\vec{A}_{1x} + \vec{A}_{2x})\,i \qquad 2.23$$

The sum of the magnitudes of x-components is given by

$$A_x = A_{1x} + A_{2x} \qquad 2.24(a)$$

$$A_x = A_1\cos\theta_1 + A_2\cos\theta_2 \qquad 2.24(b)$$

(v) Similarly the sum of component vectors along y-axis

$$\vec{A}_y = \vec{A}_{1y} + \vec{A}_{2y} \qquad 2.25$$

or

$$\vec{A}_y = (A_{1y} + A_{2y})\,j$$

The sum of the magnitudes of y-components is given by

$$A_y = A_{1y} + A_{2y} \qquad 2.26(a)$$

$$A_y = A_1\sin\theta_1 + A_2\sin\theta_2 \qquad 2.26(b)$$

(vi) The magnitude of resultant vector is

$$A = \sqrt{A_x^2 + A_y^2} \qquad 2.27(a)$$

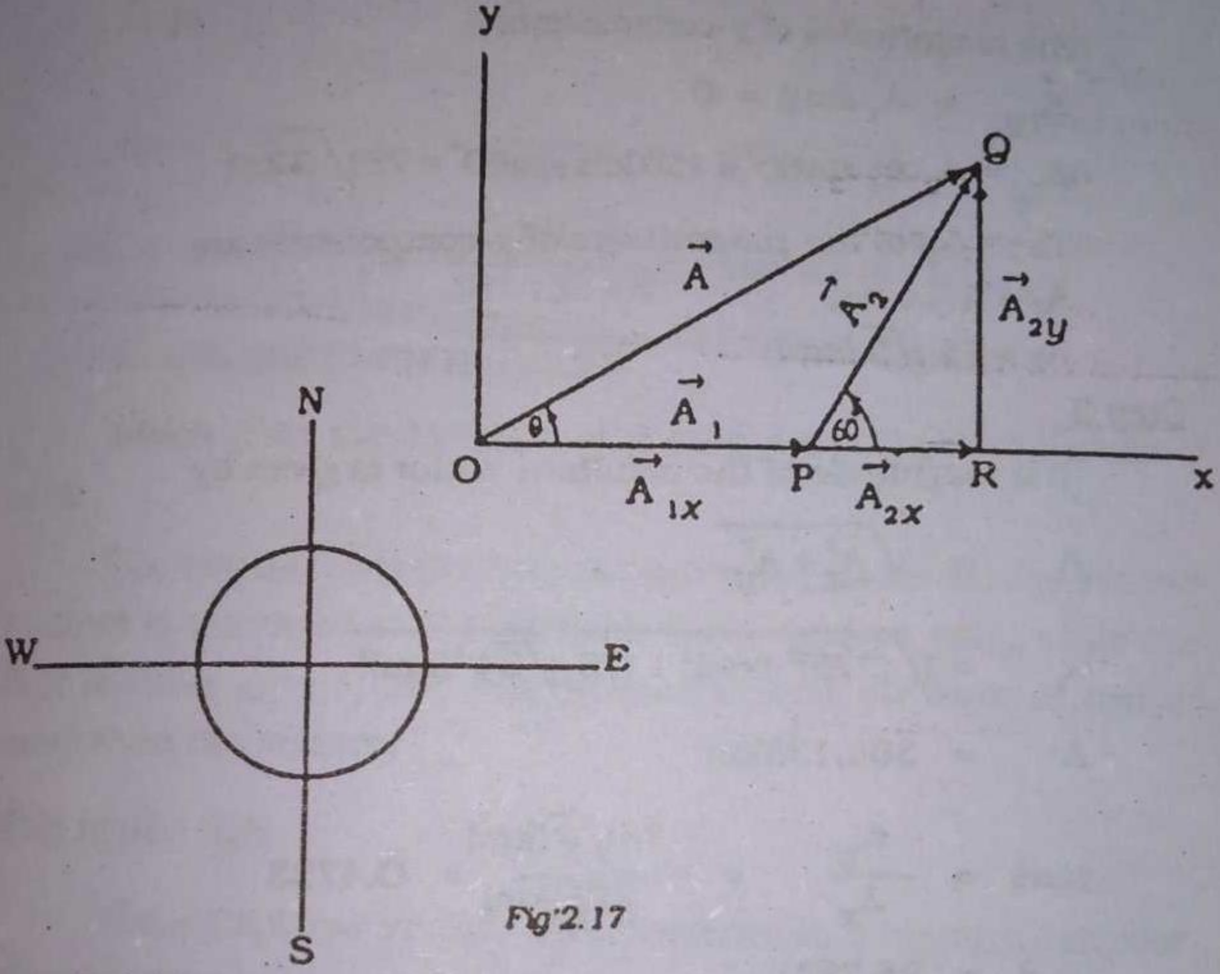
$$A = \sqrt{[A_1\cos\theta_1 + A_2\cos\theta_2]^2 + [A_1\sin\theta_1 + A_2\sin\theta_2]^2} \qquad 2.27(b)$$

The direction of resultant vector

$$\theta = \tan^{-1}\left[{A_y}/{A_x}\right] \qquad 2.28$$

Example 2.5

An automobile travels 200 km due east and then 150 km 60° north of east. Determine the resultant displacement and the direction of the resultant with respect to positive x-axis.



Fig'2.17

Solution

Method— 1

We choose the positive x-axis to be east and positive y-axis to be north direction Fig 2.17. First we draw the line OP=4cm (50km=1cm)along the +ve x-axis which represents 200 km displacement vector $\vec{A}$. Next we draw the line PQ=3cm which makes and angle of 60° with +ve x-axis measured counter clockwise, this represents 150km displacement vector $\vec{A_2}$

Step 1. The magnitudes of x-components are

$$A_{1x} = A_1 \cos 0 = 200\text{km} \quad [\because \theta = 0]$$

$$A_{2x} = A_2 \cos 60^\circ = 150\,\text{km} \cos 60^\circ = 75\text{km}$$

The sum of the magnitudes of x-components

$$A_x = A_{1x} + A_{2x}$$
$$= 275\ km$$

Step 2.

The magnitudes of y-components

$$A_{1y} = A_1 \sin\theta = 0$$
$$A_{2y} = A_2 \sin 60^\circ = 150\,km \sin 60^\circ = 75\sqrt{3}\,km$$

The sum of the magnitudes of y-components are

$$A_y = A_{1y} + A_{2y}$$
$$A_y = 75\sqrt{3}\ (km)$$

Step 3.

The magnitude of the resultant vector is given by

$$A = \sqrt{A_x^2 + A_y^2}$$
$$= \sqrt{(275)^2\,(km)^2 + (75\sqrt{3})^2 (km)^2}$$
$$A = 304.138 km$$

$$\tan\theta = \frac{A_y}{A_x} = \frac{75\sqrt{3}\,(km)}{275\,(km)} = 0.4723$$
$$\theta = 25.28^\circ$$

thus magnitude of resultant vector = 304.138 km and direction of the resultant vector is 25.28° north of east.

Method—2

Alternatively: from triangle OPQ, we have by the law of Cosines.

$$A = \sqrt{A_1^2 + A_2^2 \quad 2A_1A_2 \cos\angle OPQ}$$
$$A = \sqrt{(200)^2 (km)^2 + (150)^2 (km)^2 - 2 \times 200km \times 150\ km \cos 12}$$
$$A = 304.138km$$

Also by the law of sines

$$\frac{A}{\sin\angle OPQ} = \frac{A_1}{\sin\angle PQO} = \frac{A_2}{\sin\angle QOP}$$

$$\frac{A_2}{\sin\angle QOP} = \frac{A}{\sin\angle OPQ}$$

$$\sin\angle QOP = \frac{A_2}{A} \sin\angle OPQ$$

$$= \frac{150\text{km}}{304.138\text{ km}} \sin 120^\circ = 0.4271$$

$$\therefore \theta = \sin^{-1}(0.4271) = 25.28^\circ$$

which gives the direction of the resultant i.e; 25.28° north of east.

The two methods produce same results. However, the second method is restricted to the addition of two vectors only, while the first is more general and can be used to find the sum of two or more than two vectors.

Example 2.6

Three Coplanar vectors with reference to a rectangular coordinate system are

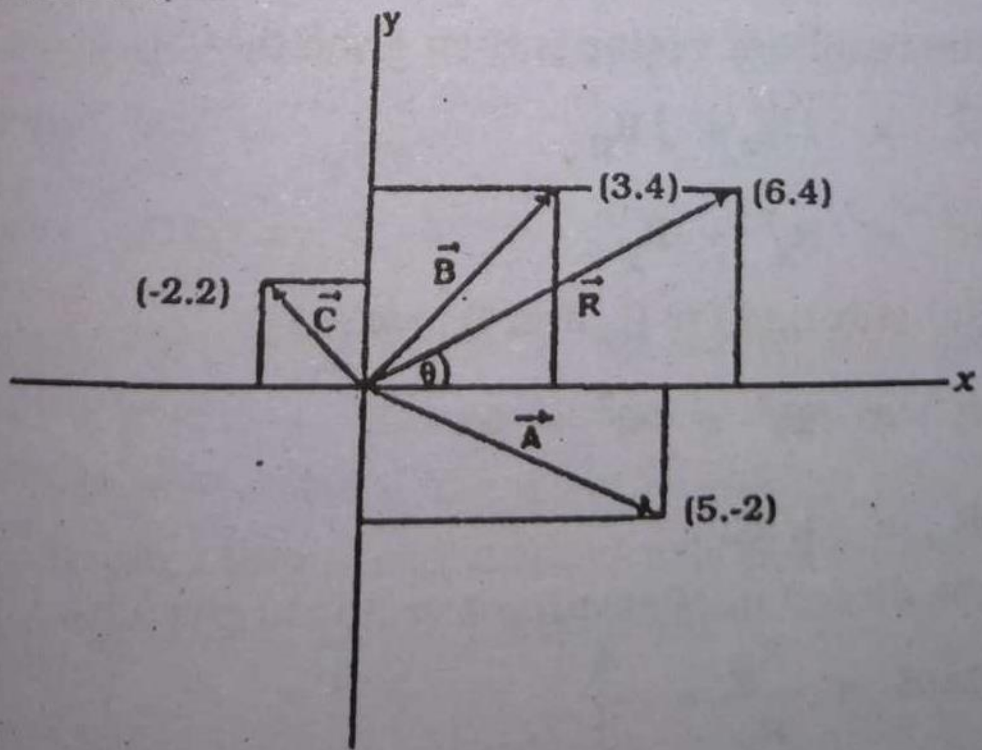


Fig. 2.18

$$\vec{A} = 5i - 2j$$
$$\vec{B} = 3i + 4j$$
$$\vec{C} = -2i + 2j$$

and the components are given by arbitrary units. Find the resultant vector $\vec{R}$ which represents the sum of these vectors.

Solution

The vectors $\vec{A}$, $\vec{B}$ and $\vec{C}$ can be expressed in terms of their components and unit vectors as

$$\vec{A} = iA_x + jA_y$$

$$\vec{B} = iB_x + jB_y$$

$$\vec{C} = iC_x + jC_y$$

then

$$R_x = A_x + B_x + C_x = 5 + 3 + (-2) = 6$$

$$R_y = A_y + B_y + C_y = -2 + 4 + 2 = 4$$

the resultant vector is then given by

$$\vec{R} = iR_x + jR_y$$

$$R^2 = R_x^2 + R_y^2$$

Substituting for R_x and R_y we get

$$R^2 = (6)^2 + (4)^2 = 52$$

$$R = \sqrt{52}$$

the direction of resultant vector is given by

$$\tan\theta = \frac{R_y}{R_x} = \frac{4}{6}$$

$$\theta = \tan^{-1} \frac{4}{6} = 33.69^\circ$$

The vectors $\vec{A}, \vec{B}, \vec{C}$, and $\vec{R}$ are drawn in Fig. 2.18 The angle θ gives the direction of the resultant vector w.r.t +ve x-axis measured counter clock wise from the +ve x-axis.

2.13 THE DOT PRODUCT

We have studied earlier the multiplication of a vector by a number. We now turn to multiplication of a vector by a vector. Like scalars, vectors of different kinds can be multiplied by one another to generate quantities of new physical dimension as explained below:

(A) SCALAR PRODUCT OF TWO VECTORS

The operation of scalar product of two vectors involves the multiplication of two given vectors in such a way that the product is a scalar.

Consider two vector $\vec{A}$ and $\vec{B}$ having magnitude A and B respectively and having angle θ between them as shown in Fig.2.19 (a). The scalar product of two vectors $\acute{A}$ and $\acute{B}$ is defined as "the product of magnitudes of the vectors and the cosine of the angle between them". Thus

$$\vec{A} . \vec{B} = AB\cos\theta ; \quad 0 \leqq \theta \leqq \pi \qquad 2.29$$

the angle θ between $\vec{A}$ and $\vec{B}$ is the smaller angle between the positive direction of $\vec{A}$ and $\vec{B}$, i.e $\theta \leqq 2\pi - \theta$, which is inequality between two possible choices. The quantity (ABcosθ) is a scalar quantity, hence the name "scalar product". The quantity ABcosθ is also called dot product of the two vectors $\vec{A}$ and $\vec{B}$.

In Particular

(i) If $\vec{A}$ is parallel to $\vec{B}$, i.e $\theta = 0°$ then

$$\vec{A}.\vec{B} = AB \qquad 2.30$$

(ii) If $\vec{A} = \vec{B}$ i.e $\vec{A}$ is parallel and equal to $\vec{B}$ then

$$\vec{A}.\vec{B} = AA = A^2 \quad \because \quad \theta = 0° \qquad 2.31$$

(iii) If $\vec{A}$ is perpendicular to $\vec{B}$, i.e $\theta = 90°$, or one of the two vectors is a null vector then

$$\vec{A}.\vec{B} = 0$$

(iv) The unit vectors $\hat{i}, \hat{j}, \hat{k}$ are perpendicular to each other therefore,

$$\hat{i}.\hat{i} = \hat{j}.\hat{j} = \hat{k}.\hat{k} = 1 \qquad 2.33$$

$$\hat{i}.\hat{j} = \hat{j}.\hat{k} = \hat{k}.\hat{i} = 0 \qquad 2.34$$

2.14 COMMUTATIVE LAW FOR DOT PRODUCT

It follows from the knowledge of projection of one vector onto the direction of another, that the scalar product of vector $\vec{A}$ and vector $\vec{B}$ is equal to the mangnitude, A, of vector $\vec{A}$ times the projection of vector $\vec{B}$ onto the direction of $\vec{A}$ as shown in Fig. 2.19(b) and vice versa as shown in Fig. 2.19(c), i.e.,

(i) From Fig: 2.19 (b)

$$\vec{A}.\vec{B} = AB_A = AB\cos\theta \qquad 2.35$$

Where B_A represents the projection of vector $\vec{B}$ onto the direction of vector $\vec{A}$

(ii) From Fig: 2.19 (c)

$$\vec{B}.\vec{A} = BA_B = BA\cos\theta = AB\cos\theta \qquad 2.36$$

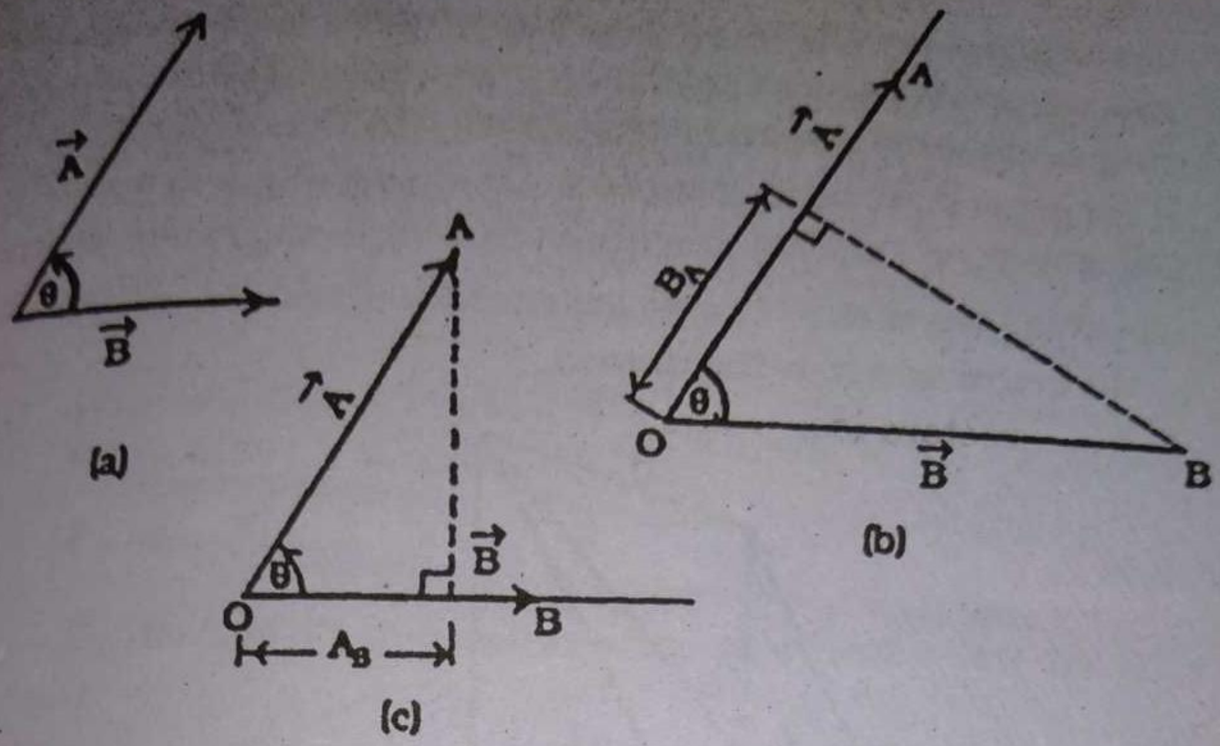


Fig. 2.19 (a) Two vectors $\vec{A}$ and $\vec{B}$ and having angle θ between them
(b) Projection of vector $\vec{B}$ onto the direction of vector $\vec{A}$
(c) Projection of vector $\vec{A}$ onto the direction of vector $\vec{B}$

where A_B represents the projection of vector $\vec{A}$ onto the direction of vector $\vec{B}$ comparing eq. 2.35 and eq. 2.36, we get

$$AB_A = BA_B$$

therefore

$$\vec{A}.\vec{B} = \vec{B}.\vec{A}$$

thus the scalar product of two vectors does not change with the change in the order of the vectors to be multiplied. Hence scalar product of two vectors obeys commutative law for dot product.

2.15 DISTRIBUTIVE LAW FOR DOT PRODUCT

To demonstrate the distributive law for dot product, we consider three vectors $\vec{A}$, $\vec{B}$ and $\vec{C}$, and we use geometrical interpretation of scalar product by drawing projection as shown in Fig 2.20. First we obtain the sum of vectors $\vec{B}$ and $\vec{C}$ by Head-to-tail rule. Then we draw the projection OC_A and OR_A from the terminal points of the vector $\vec{C}$ and the vector $(\vec{B} + \vec{C})$ respectively onto the direction of vector $\vec{A}$. The dot product $\vec{A} . (\vec{B} + \vec{C})$ is equal to the projection of the vector $(\vec{B} + \vec{C})$ onto the direction of vector $\vec{A}$ multiplied by the magnitude, A, of the vector $\vec{A}$.

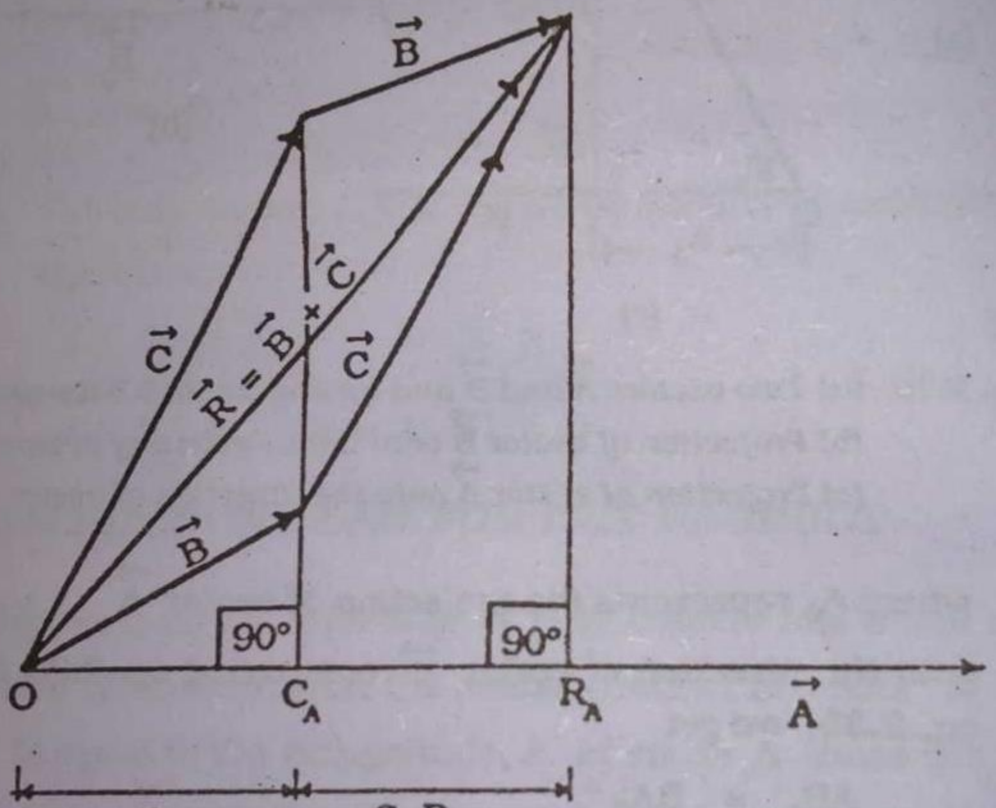


Fig. 2:20

From diagram

$$\vec{A} . (\vec{B} + \vec{C}) = A \left[OR_A \right] \qquad 2.38(a)$$

$$= A \left[C_A R_A + OC_A \right] \qquad 2.38(b)$$

$$= A \left[C_A R_A \right] + A \left[OC_A \right] \qquad 2.38(c)$$

(i) where $C_A R_A$ - represents the projection of vector $\vec{B}$ onto the direction of $\vec{A}$

(ii) OC_A - represents the projection of vector $\vec{C}$ onto the direction of vector $\vec{A}$

therefore

$$A\left[C_A R_A\right] = \vec{A}.\vec{B} \quad 2.38(d)$$

$$A\left[OC_A\right] = \vec{A}.\vec{C} \quad 2.38(e)$$

substituting for $A\left[C_A R_A\right]$ & $A\left[OC_A\right]$ in Eq.No. 2.38 (c), we get

$$\vec{A}.(\vec{B}+\vec{C}) = \vec{A}.\vec{B} + \vec{A}.\vec{C} \quad 2.38(f)$$

Eq. No.2.38 (f) demonstrates the distributive law for dot product.

Example 2.7

Evaluate the scalar product of the following:

(i) $i.i$ (ii) $i.k$ (iii) $k.(i+j)$

(iv) $(2i - j + 3k).(3i + 2j - k)$

(v) $(i - 2k).(j + 3k)$

where i, j and k represent unit vectors along x,y and z axes of three dimensional rectangular coordinate system.

Solution:

(i) $i.i = |i|\,|i| \,Cos0^\circ = 1$

(ii) $i.k = |i|\,|k|\,Cos90^\circ = 0$

(iii) $k.(i+j) = k.i + k.j$

$= |k|\,|i|\cos 90^\circ + |k|\,|j|\cos 90^\circ$

$k.(i+j) = 0$

(iv) $(2i - j + 3k).(3i + 2j - k)$

$= 6i.i + 4i.j - 2i.k - 3j.i - 2j.j + j.k +$

$9k.i + 6k.i - 3k.k$

$= 6 + 0 - 0 - 0 - 2 + 0 + 0 + 0 - 3 = 1$

(v) $(i - 2k).(j + 3k) = i.j + 3i.k - 2k.j - 6k.k$

$= 0 + 0 - 0 - 6 = -6$

Example 2.8

Find (i) the projection of $\vec{A} = 2i - 3j + 6k$ onto the direction of vector $\vec{B} = i + 2j + 2k$. (ii) determine the angle between the vectors $\vec{A}$ and $\vec{B}$.

Solution

Let a unit vector in the direction of vector $\vec{B}$ be $\hat{b}$, then by definition of a unit vector Eq.2.6

$$\hat{b} = \frac{\vec{B}}{B} = \frac{(i + 2j + 2k)}{\sqrt{(1)^2 + (2)^2 + (2)^2}} = \frac{i + 2j + 2k}{3}$$

$$\hat{b} = \frac{i}{3} + \frac{2j}{3} + \frac{2k}{3}$$

projection of vector $\vec{A}$ onto the direction of vector $\hat{b}$ is

$$\vec{A}.\hat{b} = (2i - 3j + 6k).\left(\frac{i}{3} + \frac{2j}{3} + \frac{2k}{3}\right)$$

$$= \frac{2}{3} - \frac{6}{3} + \frac{12}{3} = \frac{8}{3}$$

$\vec{A}.\hat{b} = 8/3$

also

$\vec{A}.\hat{b} = |\hat{b}|\,|\vec{A}| \cos\theta$

$$\cos\theta = \frac{\vec{A}.\hat{b}}{\hat{b}\,|\vec{A}|} = \frac{8/3}{|\vec{A}|}$$

$A = |\vec{A}| = \sqrt{(2)^2 + (-3)^2 + (6)^2} = 7$

$$\cos\theta = \frac{8/3}{7} = 8/21$$

$$\theta = 67.6^\circ$$

Example 2.9

Find the work done in moving an object along a straight line from (3,2,-1) to (2,-1,4) in a force field which is given by $\vec{F} = 4i - 3j + 2k$ and also find the angle between force and displacement.

Solution

Let W represent work which is given by.

$$W = \vec{F} \cdot \vec{d}$$

where $\vec{F}$ is applied force

$\vec{d}$ is displacement which is given by

$$\vec{d} = (x_2 - x_1)\,i + (y_2 - y_1)\,j + (z_2 - z_1)\,k$$

$$= (2 - (+3))i + (-1-(+2)\,j + (4 - (-1))\,k$$

$$= (-i) + (-3j) + (5k)$$

$$\vec{d} = (-i) + (-3j) + (5k)$$

$$W = \vec{F} \cdot \vec{d} = (4i - 3j + 2k) \cdot (-i - 3j + 5k)$$

$$= -4i.i + 9j.j + 10k.k$$

$$W = -4 + 9 + 10 = 15$$

By definition

$$W = \vec{F} \cdot \vec{d} = Fd\cos\theta$$

Where θ is the angle between $\vec{F}$ and $\vec{d}$

$$W = Fd\cos\theta$$

to find out the angle θ we need to know the magnitudes of the vectors $\vec{F}$ and $\vec{d}$

$$F = \sqrt{(4)^2 + (-3)^2 + (2)^2} = \sqrt{29}$$

$$d = \sqrt{(-1)^2 + (-3)^2 + (5)^2} = \sqrt{35}$$

substituting for W.F and d. we get

$$\cos\theta = 15/(\sqrt{29} \times \sqrt{35})$$

$$= 0.47$$

$$\theta = 61.91°$$

Example 2.10

Two vectors $\vec{A}$ and $\vec{B}$ are such that $|\vec{A}| = 4$, $|\vec{B}| = 6$ and $\vec{A}.\vec{B} = 13.5$. Find the magnitude of vector $|\vec{A} - \vec{B}|$ and the angle between $\vec{A}$ and $\vec{B}$.

Solution

$$\because |\vec{A}| = \sqrt{\vec{A}.\vec{A}}$$

$$|\vec{A} - \vec{B}| = \sqrt{(\vec{A} - \vec{B}).(\vec{A} - \vec{B})}$$

$$= \sqrt{\vec{A}.\vec{A} - 2\vec{A}.\vec{B} + \vec{B}.\vec{B}}$$

The expression $\vec{A}.\vec{A} = |\vec{A}|^2 = A^2$.

Similarly $\vec{B}.\vec{B} = |\vec{B}|^2$, then

$$|\vec{A} - \vec{B}| = \sqrt{|\vec{A}|^2 \quad 2\vec{A}.\vec{B} + |\vec{B}|^2}$$

$$= \sqrt{(4)^2 - 2 \times 13.5 + (6)^2} = 5$$

Also

$$\vec{A}.\vec{B} = |\vec{A}|\ |\vec{B}| \cos\theta$$

$$\cos\theta = \frac{\vec{A}\cdot\vec{B}}{|\vec{A}||\vec{B}|} = \frac{13.5}{4\times6} = 13.5/24$$

$$\theta = 55.77^\circ$$

2.16 THE CROSS PRODUCT

In preceding sections, we developed and discussed the concept of multiplication of two vectors in such a way that their resultant product is a scalar quantity. When dealing with quantities such as torque, angular momentum, the force on a moving charge in a magnetic field, flow of electro magnetic energy, etc, we turn to the multiplication of two given vectors in such a way that the resultant product is a vector quantity. This product is known as vector product or cross product.

Consider two vectors $\vec{A}$ and $\vec{B}$, the vector product of these two vectors is denoted by $\vec{A}\times\vec{B}$, and read as "$\vec{A}$ cross $\vec{B}$". The cross or vector product of $\vec{A}$ and $\vec{B}$, is a new vector $\vec{C} = \vec{A}\times\vec{B}$, by definition the vector $\vec{C}$ is perpendicular to the plane containing the vectors $\vec{A}$ and $\vec{B}$. By definition (i) the magnitude, $|\vec{A}\times\vec{B}|$, of the cross product or the magnitude, $|\vec{C}|$, of the vector $\vec{C}$ is given by

$$C = |\vec{C}| = AB\sin\theta,\ 0 \leq \theta \leq \pi \qquad 2.39$$

Where $\vec{A}$ and $\vec{B}$, represent the magnitudes of vectors $\vec{A}$ and $\vec{B}$ respectively. θ is smaller angle between the positive direction of $\vec{A}$ and $\vec{B}$, i.e. $\theta \leq 2\pi - \theta$.

(ii) The vector $\vec{C} = \vec{A}\times\vec{B}$, which represents the cross or vector product is perpendicular to the plane containing vectors $\vec{A}$ and $\vec{B}$ (by definition) and points in the direction in such a way as to make $\vec{A}$, $\vec{B}$, and $\vec{C}$, in that order, a right handed system as shown in Fig.2.21 (a).

We generalize this definition and write

$$\vec{C} = \vec{A}\times\vec{B} = [AB\sin\theta]\,\hat{u} \qquad 2.40$$

Where $\hat{u}$ is a unit vector perpendicular to both $\vec{A}$ and $\vec{B}$ and in the sense determined by a right handed screw turning from $\vec{A}$ to $\vec{B}$.

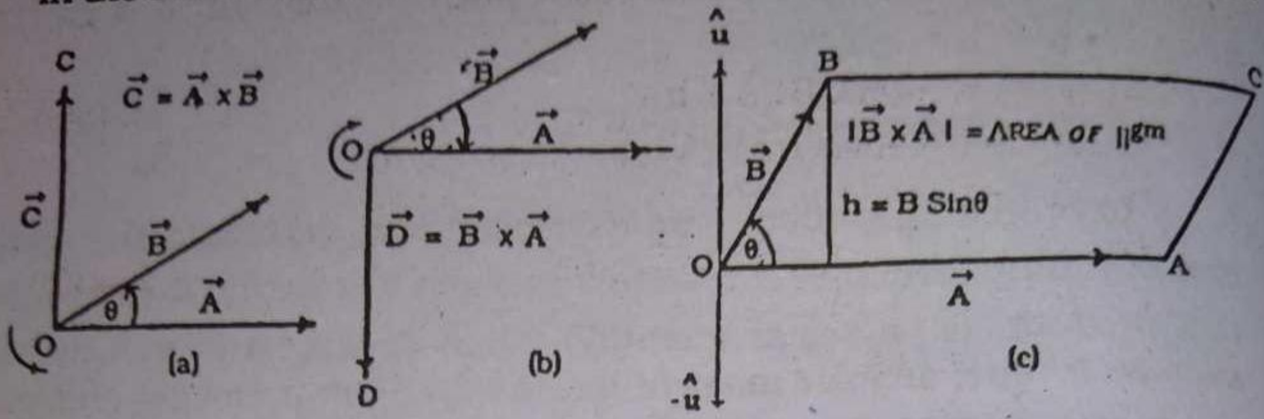


Fig. 2.21 shows vector product $\vec{A} \times \vec{B}$ In right handed coordinate system.
Fig. 2.21(a) shows the direction of the vector $\vec{C}$ is that in which a right handed screw advances when turned from $\vec{A}$ to $\vec{B}$
Fig 2.21 (b) shows the direction of the vector $\vec{D}$ changes through 180° when turned from $\vec{B}$ to $\vec{A}$.
Fig. 2.21 (c)The area of Parallelogram is given by the magnitude of the cross product

Similarly a right handed screw turning from $\vec{B}$ to $\vec{A}$ defines the unit vector $-\hat{u}$. then

$$\vec{D} = (\vec{B} \times \vec{A}) = [B\,A \sin\theta]\,(-\hat{u}) \qquad 2.41(a)$$

$$= (\vec{B} \times \vec{A}) = -[B\,A \sin\theta]\,(\hat{u})$$

$$-\vec{D} = -(\vec{B} \times \vec{A}) = [B\,A \sin\theta]\,\hat{u} \qquad 2.41(b)$$

The quantities AB sin θ and BA sin θ on the R.H.S. of Eq 2.40 and 2.41 (b) being the magnitudes are equal therefore

$$\vec{A} \times \vec{B} = -\vec{B} \times \vec{A} \text{ or } \vec{C} = -\vec{D} \qquad 2.42$$

The Eq: 2.42 signifies that the commutative law for cross product is not valid.

$$\vec{A} \times \vec{B} = -\vec{B} \times \vec{A} \text{ or } \vec{C} = -\vec{D}$$

The Eq. 2.42 signifies that vector multiplication is not commutative.

To give physical interpretation of vector product of two given vectors, we consider two vectors $\vec{A}$ and $\vec{B}$. Let these vectors represent two adjacent sides of a parallelogram OBCA as shown in Fig.2.21 (c). From figure, the area of the parallelogram OBCA is given by

the area of llgm OBCA = hA 2.43(a)

If the angle between two vectors is θ, then

$h = B \sin\theta$

the area of llgm OBCA = $A\, B \sin\theta$ 2.43(b)

By definition we know

(i) $C = AB\sin\theta$ 2.44

and

$$\vec{C} = \vec{A} \times \vec{B} \qquad 2.45$$

(ii) the vector product. $A \times \vec{B}$, is perpendicular to the plane containing both $\vec{A}$ and $\vec{B}$ Comparing eq. 2.43(b) and Eq 2.44, we conclude that the cross product is perpendicular to the parallelogram defined by vector $\vec{A}$ and Vector $\vec{B}$ and its magnitude is equal to the area of the parallelogram.

2.17 SOME PHYSICAL EXAMPLES OF VECTOR PRODUCT.

(i) The simplest example of a vector product is the moment M of a force about a point O, defined as

$$\vec{M} = \vec{R} \times \vec{F}$$

Where $\vec{R}$ is a vector joining the point 'O' to the initial point of $\vec{F}$.

(ii) An electric charge, q, moving with velocity $\vec{V}$ in a magnetic field $\vec{B}$ experiences a force $\vec{F}$, which is given by

$$\vec{F} = q(\vec{V} \times \vec{B}).$$

2.18 PROPERTIES OF THE VECTOR PRODUCT

The following are the important properties of vector product:

(i) $\vec{A} \times \vec{B} = -(\vec{B} \times \vec{A})$ 2.46

(ii) $\vec{A} \times (\vec{B} + \vec{C}) = (\vec{A} \times \vec{B}) + (\vec{A} \times \vec{C})$ 2.47

(iii) $(\vec{A} + \vec{B}) \times \vec{C} = (\vec{A} \times \vec{C}) + (\vec{B} \times \vec{C})$ 2.48

(iv) If $\vec{A} \neq 0, \vec{B} \neq 0$ and $\vec{A} \times \vec{B} = o$, then

$\vec{A}$ and $\vec{B}$ are parallel.

(v) $i \times i = 0$ 2.49

$j \times j = 0$ 2.50

$k \times k = 0$ 2.51

(vi) $i \times j = k$ 2.52

$j \times k = i$ 2.53

$k \times i = j$ 2.54

(vii) $i \times j = -j \times i = k$ 2.55(a)

$j \times k = -k \times j = i$ 2.56(b)

$k \times i = -i \times k = j$ 2.57(c)

(viii) If $\vec{A} = A_1 i + A_2 j + A_3 k$

$\vec{B} = B_1 i + B_2 j + B_3 k$

the cross product or vector product will be written as

$$\vec{A} \times \vec{B} = \begin{vmatrix} i & j & k \\ A_1 & A_2 & A_3 \\ B_1 & B_2 & B_3 \end{vmatrix}$$

Example 2.11

$\vec{R}_1$ and $\vec{R}_2$ are two position vectors making angle θ_1 and θ_2 with positive x-axis respectively. Find their vector product when

$R_1 = 4$cm, $R_2 = 3$ cm

$\theta_1 = 30°, \theta_2 = 90°$

Solution

The angle between two vectors is $\theta = \theta_2 - \theta_1 = 60°$ the magnitude of the cross product of vectors $\vec{R}_1$ and $\vec{R}_2$ is

$$C = R_1 R_2 \sin\theta$$

$$= 4 \times 3 \times \frac{\sqrt{3}}{2} = 6\sqrt{3}$$

As the vectors $\vec{R}_1$ and $\vec{R}_2$ lie in x-y plane, therefore the vector representing cross product lie parallel to z-direction.

Example 2.12

Two sides of a triangle are formed by the vector $\vec{A} = 3i + 6j - 2k$ and vector $\vec{B} = 4i - j + 3k$. Determine the area of the triangle.

Solution

The area of triangle in terms of vector product is given by $\frac{1}{2}|(\vec{A} \times \vec{B})|$.

$$\vec{A} \times \vec{B} = (3i + 6j - 2k) \times (4i - j + 3k)$$

$$= 3i \times (4i - j + 3k) + 6j \times (4i - j + 3k) - 2k \times (4i - j + 3k)$$

$$= 12i \times i - 3i \times j + 9i \times k + 24j \times i - 6j \times j + 18j \times k - 8k \times i + 2k \times j - 6k \times k$$

$$\vec{A} \times \vec{B} = 16i - 17j + 27k$$

$$C = |\vec{A} \times \vec{B}| = \sqrt{(16)^2 + (-17)^2 + (-27)^2} = \sqrt{1274}$$

$$\frac{1}{2}C = \frac{1}{2}\sqrt{1274}$$

which is the required area of triangle Alternatively.

$$\vec{A} \times \vec{B} = (3i + 6j - 2k) \times (4i - j + 3k)$$

$$\vec{A} \times \vec{B} = \begin{vmatrix} i & j & k \\ 3 & 6 & -2 \\ 4 & -1 & 3 \end{vmatrix}$$

$$= i \begin{vmatrix} 6 & -2 \\ -1 & 3 \end{vmatrix} + j \begin{vmatrix} -2 & 3 \\ 3 & 4 \end{vmatrix} + k \begin{vmatrix} 3 & 6 \\ 4 & -1 \end{vmatrix}$$

$$= i(18-2) + j(-8-9) + k(-3-24)$$

$$= 16i - 17j - 27k$$

$$C = \sqrt{(16)^2 + (-17)^2 + (-27)^2} = \sqrt{1274}$$

$$\frac{1}{2} C = \frac{1}{2}\sqrt{1274}$$

Example 2.13

Determine a unit vector perpendicular to the plane containing $\vec{A}$ and $\vec{B}$, if $\vec{A} = 2i - 3j - k$, $\vec{B} = i + 4j - 2k$

Solution

By definition $\vec{A} \times \vec{B}$ is vector which is perpendicular to the plane containing $\vec{A}$ and $\vec{B}$ and

$$\vec{C} = \vec{A} \times \vec{B} = (2i - 3j - k) \times (i + 4j - 2k)$$

$$\vec{A} \times \vec{B} = \begin{vmatrix} i & j & k \\ 2 & -3 & -1 \\ 1 & 4 & -2 \end{vmatrix}$$

$$= i \begin{vmatrix} -3 & -1 \\ 4 & -2 \end{vmatrix} + j \begin{vmatrix} -1 & 2 \\ -2 & 1 \end{vmatrix} + k \begin{vmatrix} 2 & -3 \\ 1 & 4 \end{vmatrix}$$

$$= i(6+4) + j(-1+4) + k(8+3)$$

$$= 10i + 3j + 11k$$

A unit vector parallel to $\vec{A} \times \vec{B}$ is given by Eq 2.6

$$\hat{C} = \frac{\vec{C}}{C} = \frac{10i + 3j + 11k}{\sqrt{(10)^2 + (3)^2 + (11)^2}}$$

$$\hat{C} = \frac{10i + 3j + 11k}{\sqrt{230}}$$

$$\hat{C} = \frac{\vec{C}}{C} = \frac{10i}{\sqrt{230}} + \frac{3j}{\sqrt{230}} + \frac{11k}{\sqrt{230}}$$

PROBLEMS:-

1. State which of the following are scalars and which are vectors.

(1) Weight
(2) Calorie
(3) Specific heat
(4) Momentum
(5) Density
(6) Energy
(7) Volume
(8) Distance
(9) Speed
(10) Magnetic field intensity
(11) Entropy
(12) Work
(13) Centrifugal force
(14) temperature
(15) gravitational potential
(16) Charge
(17) Shearing stress
(18) frequency
(19) Kinetic energy
(20) Electric field intensity.

Ans:

(1) vector	(2) scalar	(3) scalar
(4) vector	(5) scalar	(6) scalar
(7) scalar	(8) scalar	(9) scalar
(10) vector	(11) scalar	(12) scalar
(13) vector	(14) scalar	(15) scalar
(16) scalar	(17) vector	(18) scalar
(19) scalar	(20) vector	

2. Find the resultant of the following displacement:

$\vec{A}$ = 20 Km 30° south of east;

$\vec{B}$ = 50km due west

$\vec{C}$ = 40 km north east;

$\vec{D}$ = 30 Km 60° south of west.

Ans: Magnitude 20.9 km. direction 21.65° south of west

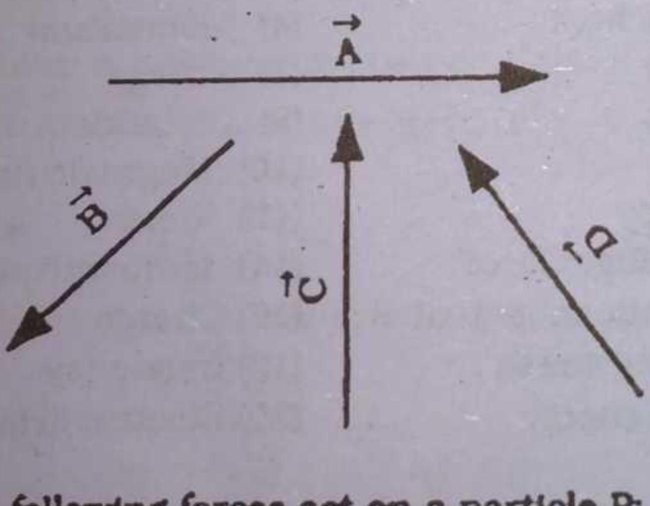
3. An aeroplane flies 400 km due west from city A to city B, then 300 km north east to city C, and finally 100 km north to cityD. How far is it from city A to D? In what direction must the aeroplane had to return directly to city A from city D?

Ans: 364 km at 31° east of south.

4. Show graphically that $-(\vec{A} - \vec{B}) = -\vec{A} + \vec{B}$

5. Given vector $\vec{A}$, $\vec{B}$, $\vec{C}$ and $\vec{D}$ as shown in figure below. Construct (a) $4\vec{A} - 3\vec{B} - (2\vec{C} + 2\vec{D})$

(b) $(1/2)(\vec{C}) + (1/3)(\vec{A} + \vec{B} + 2\vec{D})$



6. The following forces act on a particle P:

$\vec{F}_1 = 2i + 3j - 5k$, $\vec{F}_2 = -5i + j + 3k$,

$\vec{F}_3 = i - 2j + 4k$, $\vec{F}_4 = 4i - 3j - 2k$,

measured in newtons Find (a) the resultant of the forces

(b) the magnitude of the resultant force

Ans. (a) $2i - j$ (b) $\sqrt{5}$

7. If $A = 3\hat{i} - \hat{j} - 4\hat{k}$, $B = -2\hat{i} + 4\hat{j} - 3\hat{k}$
$C = \hat{i} + 2\hat{j} - \hat{k}$, find

(a) $2\vec{A} - \vec{B} + 3\vec{C}$, (b) $|\vec{A} + \vec{B} + \vec{C}|$.

(c) $|3\vec{A} - 2\vec{B} + 4\vec{C}|$

(d) a unit vector parallel to $3\vec{A} - 2\vec{B} + 4\vec{C}$.

Ans. (a) $9i - 4j - 6k$ (b) $\sqrt{93}$ (c) $\sqrt{398}$ (d) $(3\vec{A} - 2\vec{B} + 4\vec{C})/\sqrt{398}$

8. Two tugboats are towing a ship. Each exerts a force of 6000N, and the angle between the two ropes is $60°$. Calculate the resultant force on the ship.

Ans. 10392 N

9. The position vectors of points P and Q are given by $\vec{r}_1 = 2i + 3j - k$, $\vec{r}_2 = 4i - 3j + 2k$. Determine $\overrightarrow{PQ}$ in terms of rectangular unit vector **i**, **j** and **k** and find its magnitude.

Ans. $2i - 6j + 3k$, 7

10. Prove that the vectors $\vec{A} = 3i + j - 2k$,

$\vec{B} = -i + 3j + 4k$, $\vec{C} = 4i - 2j - 6k$,

can form the sides of a triangle. Find the length of the medians of the triangle.

Ans: $\sqrt{6}$, $\frac{1}{2}\sqrt{114}$, $\frac{1}{2}\sqrt{150}$

11. Find the rectangular components of a vector $\vec{A}$, 15 unit long when it form an angle with respect to +ve x-axis of (i) $50°$, (ii) $130°$ (iii) $230°$, (iv) $310°$

Ans. (i) A_x = 9.6 unit, A_y = 11.5 unit

(ii) A_x = -9.6 unit, A_y = 11.5 unit

(iii) $A_x = -9.6$ unit. $A_y = -11.5$ unit.

(iv) $A_x = 9.6$ unit $A_y = -11.5$ unit.

12. Two vectors 10cm and 8cm long form an angle of (a) 60°, (b) 90° and (c) 120°. Find the magnitude of difference and the angle with respect to the larger vector.

Ans. 9.2 cm, 49° (b) 12.8 cm, 38° 41′ (c) 15.6 cm, 26°.22′

13. The angle between the vector $\vec{A}$ and $\vec{B}$ is 60°. Given that $|\vec{A}| = |\vec{B}| = 1$, calculate (a) $|\vec{B}-\vec{A}|$; (b) $|\vec{B}+\vec{A}|$

Ans (a) 1, (b) $\sqrt{3}$

14. A car weighing 10,000 N on a hill which makes an angle of 20° with the horizontal. Find the components of car's weight parallel and perpendicular to the road.

Ans. $F_{\parallel} = 3420$ N, $F_{\perp} = 9400$N

15. Find the angle between $A = 2i + 2j - k$ and $B = 6i - 3j + 2k$.

Ans. $\theta = 79°$

16. Find the projection of the vector. $A = i - 2j + k$ onto the direction of vector $B = 4i - 4j + 7k$.

Ans. 19/09

17. Find the angles α, β, γ which the vector $A = 3i - 6j + 2k$ makes with the positive x,y,z axis respectively.

Ans. $\alpha = 64.6°$, $\beta = 149°$, $\gamma = 73.4°$

18. Find the work done in moving an object along a vector $\vec{r} = 3i + 2j - 5k$ if the applied force is $\vec{F} = 2i - j - k$.

Ans.9

19. Find the work done by a force of 30.000N in moving an object through a distance of 45m when: (a) the force is in the direction of motion; and (b) the force makes an angle of 40° to the direction of motion. Find the rate at which the force is working at a time when the velocity is 2m/sec.

Ans. (a) 1.35×10^{6} J, 6×10^{4} w (b) 1.03×10^{6} J, 4.60×10^{4} w.

20. Two vectors $\vec{A}$ and $\vec{B}$ are such that $|\vec{A}| = 3$, $|\vec{B}| = 4$, and $\vec{A} . \vec{B} = -5$, find

(a) the angle between $\vec{A}$ and $\vec{B}$

(b) the length $|\vec{A} + \vec{B}|$ and $|\vec{A} - \vec{B}|$

(c) the angle between $(\vec{A} + \vec{B})$ and $(\vec{A} - \vec{B})$

Ans. (a) 114.6° (b) $\sqrt{15}$, $\sqrt{35}$ (c) 107°.8

21. If $\vec{A} = 2i - 3j - k$, $\vec{B} = i + 4j - 2k$.
Find (a) $\vec{A} \times \vec{B}$. (b) $\vec{B} \times \vec{A}$. (c) $(\vec{A} + \vec{B}) \times (\vec{A} - \vec{B})$
Ans. (a) $10i + 3j + 11k$ (b) $-10i - 3j - 11k$
(c) $-20i - 6j - 22k$

22. Determine the unit vector perpendicular to the plane of $\vec{A} = 2i - 6j - 3k$ and $\vec{B} = 4i + 3j - k$

Ans. $+(\frac{3}{7}i \quad \frac{2}{7}j + \frac{6}{7}k)$

23. Using the definition of vector product, prove the law of sines for plane triangles of sides a, b and c.

Ans. $\sin A/a = \sin B/b = \sin C/c$

24. If $\vec{r}_1$ and $\vec{r}_2$ are the position vectors (both lie in xy plane) making angle θ_1 and θ_2 with the positive x - axis measured counter clockwise, find their vector product when

(i) $|\vec{r}_1| = 4$ cm $\quad \theta_1 = 30^{\circ}$
$|\vec{r}_2| = 3$ cm $\quad \theta_2 = 90^{\circ}$

(II) $|\vec{r}_1| = 6$ cm $\theta_1 = 220°$

$|\vec{r}_2| = 3$ cm $\theta_2 = 40°$

(III) $|\vec{r}_1| = 10$ cm $\theta_1 = 20°$

$|\vec{r}_2| = 9$ cm $\theta_2 = 110°$

Ans. (I) $6\sqrt{3}$ cm^2 (II) 0, (III) 90 cm^2