Unit 9

CONGRUENT TRIANGLES

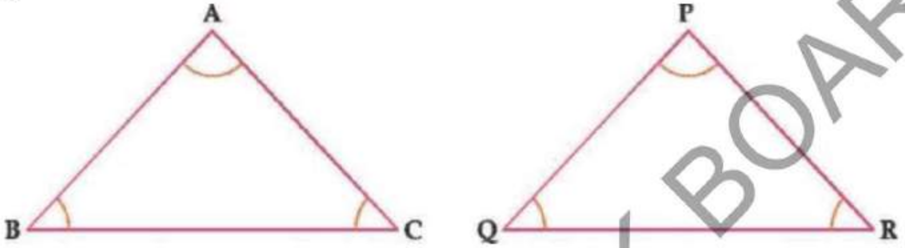
Student Learning Outcomes (SLOs)

After completing this unit, students will be able to:

- Understand the following theorems along with their corollaries and apply them to solve allied problems.
- In any correspondence of two triangles, if one side and any two angles of one triangle are congruent to the corresponding sides and angles of the other, the two triangles are congruent.
- If two angles of a triangle are congruent then the sides opposite to them are also congruent.
- In the correspondence of the two triangles, if three sides of one triangle are congruent to the corresponding three sides of the other, the two triangles are congruent or similar triangles.
- If in the correspondence of two right angled triangles, the hypotenuse and one side of one are congruent to the hypotenuse and the corresponding side of the other, then the triangles are congruent.

Introduction

A triangle has six elements, three sides and three angles. If we are given two triangles ABC and PQR, we can associate their vertices to establish a (1-1) correspondence between the sides and angles of these triangles in six different way given as under:



In the correspondence $\Delta ABC \leftrightarrow \Delta PQR$ it means

(i) $\angle A \leftrightarrow \angle P$ ($\angle A$ corresponds to $\angle P$).
(ii) $\angle B \leftrightarrow \angle Q$ ($\angle B$ corresponds to $\angle Q$).
(iii) $\angle C \leftrightarrow \angle R$ ($\angle C$ corresponds to $\angle R$).
(iv) $\overline{AB} \leftrightarrow \overline{PQ}$ ($\overline{AB}$ corresponds to $\overline{PQ}$).
(v) $\overline{BC} \leftrightarrow \overline{QR}$ ($\overline{BC}$ corresponds to $\overline{QR}$).
(vi) $\overline{CA} \leftrightarrow \overline{RP}$ ($\overline{CA}$ corresponds to $\overline{RP}$).

9.1 Congruent triangles

"Sameness of size and shape" in the mathematics called congruence.

Two cars have different colours and their position is different in the given figures. But they have same size and shape. These two cars are said to be congruent. If we keep the picture of one car on the other car then they two will overlap each other.

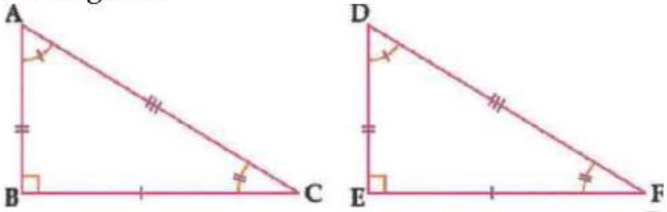
ACTIVITY

Exploration

Can you identify any congruent figures or objects in your classroom or school? Make a list of these congruent figures by drawing or taking photos.

Two triangles are said to be congruent if their corresponding angles and sides are congruent.

Let's see the figures.



These two triangles ABC and DEF are congruent and written as:

$$\Delta ABC \cong \Delta DEF$$

The ΔABC and ΔDEF having their corresponding sides and angles are equal, in measure.

Note: Following results are useful.

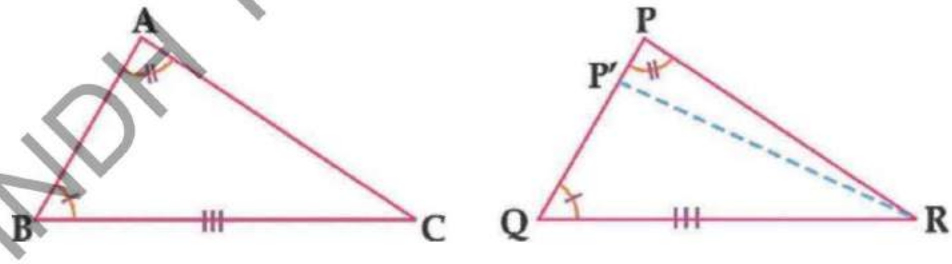
(i) Identity congruence i.e $\Delta ABC \cong \Delta ABC$.

(ii) Symmetric property i.e $\Delta ABC \cong \Delta PQR$ then $\Delta PQR \cong \Delta ABC$.

(iii) Transitive property of congruence, if $\Delta ABC \cong \Delta PQR$ and $\Delta PQR \cong \Delta DEF$, then $\Delta ABC \cong \Delta DEF$.

Theorem 9.1.1 (A.S.A. ≅ A.S.A.)

In any correspondence of two triangles, if one side and any two angles of one triangle are congruent to the corresponding sides and angles of the other, the two triangles are congruent.



Given:

In $\Delta ABC \leftrightarrow \Delta PQR$, then

$\angle B \cong \angle Q$, $m\overline{BC} \cong m\overline{QR}$,

and $\angle A \cong \angle P$.

To prove:

$\Delta ABC \cong \Delta PQR$

Construction:

Suppose, $\overline{AB} \ncong \overline{PQ}$ take a point P′ on PQ such that AB ≅ P′Q.
Join P′ to R.

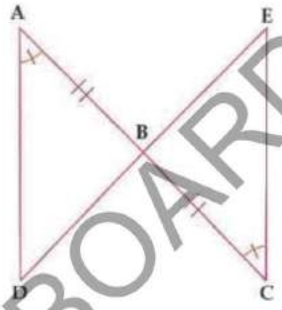
Proof:

Statements	Reasons
In $\Delta ABC \leftrightarrow \Delta PQR$	
i. $\angle A \cong \angle P$	i. Given
ii. $\angle B \cong \angle Q$	ii. Given
$\therefore \angle C \cong \angle R$	Two angles of both triangles are congruents.
If $\overline{BA} \ncong \overline{QP}$, take a point P′ on $\overline{QP}$ (or $\overline{QP}$ produced) such that: $\overline{QP'} \cong \overline{BA}$	Assumption
In $\Delta ABC \leftrightarrow \Delta P'QR$	
$\overline{BC} \cong \overline{QR}$	i. Given
$\angle B \cong \angle Q$	ii. Given
$\overline{BA} \cong \overline{QP'}$	iii. By supposition
$\therefore \Delta ABC \cong \Delta P'QR$	S.A.S postulate
$\therefore \angle C \cong \angle QRP'$	By the congurance of Δs.
But $\angle C \cong \angle QRP$	Proved in 2 (above).
$\therefore \angle QRP' \cong \angle QRP$	Transitive property of congurance
This is possible only when pointsP′ P coincide and	By angle construction postulate
$\overline{RP'} \cong \overline{RP}$	
Hence $\overline{BA} \cong \overline{QP}$	As P and P′ coincide.
In $\Delta ABC \leftrightarrow \Delta PQR$	
i. $\overline{BC} \cong \overline{QR}$	i. Given
ii. $\angle B \cong \angle Q$	ii. Given
iii. $\overline{BA} \cong \overline{QP}$	iii. Proved above
$\therefore \Delta ABC \cong \Delta PQR$	S.A.S Postulate.

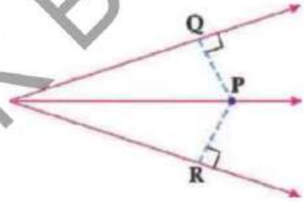
Q.E.D.

Exercise 9.1

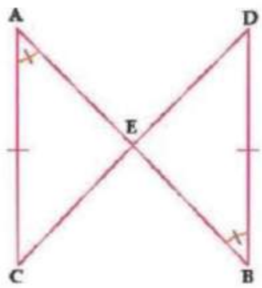
1. In the given figure, $m\overline{AB} = m\overline{CB}$ and $\angle A \cong \angle C$ prove that $\Delta ABD \cong \Delta CBE$



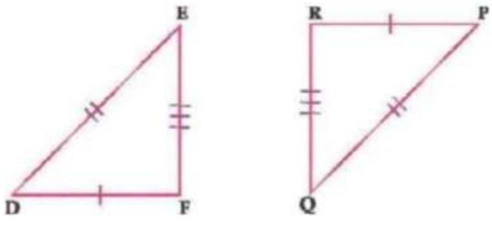
2. From a point on the line bisector of an angle, perpendiculars are drawn to the arms of the angle. Prove that these perpendiculars are equal in measure.



3. In the given figure, we have, $\Delta ACE \cong \Delta BDE$, such that:
$m\overline{AC} = m\overline{BD} = 3\ cm$,, $\angle A = (3x+1)^\circ$, $m\angle E = (3y-2)^\circ$ and
$m\angle B = (x+35)^\circ$. Find the values of x and y.

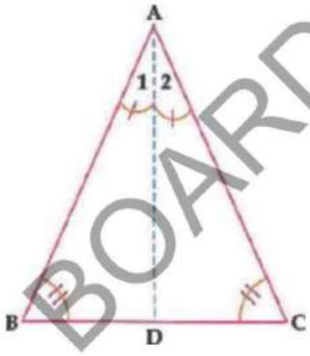


4. In the given figure,
$\Delta DEF \cong \Delta PQR$, such that:
$m\overline{DE}$= $(6x+1)$cm, $m\overline{EF}$ =8cm,
and $m\ \overline{RQ}$= $(5y-7)$cm
and $m\overline{PQ}$ = $(10x-19)$cm.
Find the values of x and y.



Theorem 9.1.2

If two angles of a triangle are congruent, then the sides opposite to them are also congruent.



Given:

In ΔABC,

We have, $\angle B \cong \angle C$

To prove:

$\overline{AB} \cong \overline{AC}$

Construction: Draw $\overline{AD}$ the bisector of $\angle A$, meeting $\overline{BC}$ at point D.

Proof:

Statement	Reason
In $\Delta ADE \leftrightarrow \Delta ADC$	
i. $\overline{AB} \cong \overline{AC}$	i. Given
ii. $\angle 1 \cong \angle 2$	ii. Construction
i. $\overline{AD} \cong \overline{AD}$	iii. Common (Identity congruence)
$\therefore$ $\Delta ABD \cong \Delta ADC$	S.A.S Postulate.
$\therefore$ $m\angle B = m\angle C$	By the congruence of Δs.

Q.E.D

Exercise 9.2

1. ABC is a triangle in which $m\angle A=35°$ and $m\angle B=100°$, $\overline{BD} \perp \overline{AC}$. Prove that ΔBDC is an isosceles triangle.

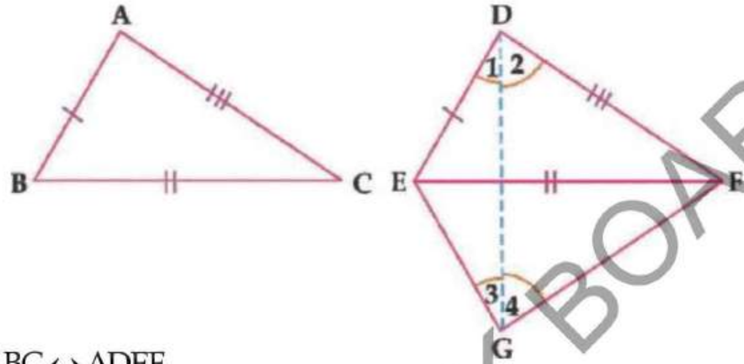
2. If the bisector of an angle of a triangle is perpendicular to its opposite side, then the triangle is an isosceles triangle.

3. ABC is a triangle in which $m\angle A = 25°$, $m\angle B = 45°$ and $\overline{CD} \perp \overline{AB}$. Prove that ΔDBC is an isosceles Δ.

Theorem 9.1.3

In a correspondence of two triangles, if three sides of one triangle are congruent to the corresponding three sides of the other, the two triangles are congruent.

Proof:



Given:

In $\Delta ABC \leftrightarrow \Delta DEF$

$\overline{AB} \cong \overline{DE}, \overline{BC} \cong \overline{EF}$ and $\overline{CA} \cong \overline{FD}$

To prove that: $\Delta ABC \leftrightarrow \Delta DEF$

Construction: Suppose $\overline{BC}$ greatest of all the three sides of ΔABC. Construct ΔGEF such that:

i. Point G
ii. $\angle FEG \cong \angle B$
iii. $\overline{EG} \cong \overline{BA}$
Join D and G.

Proof:

Statements	Reasons
In $\Delta ABC \leftrightarrow \Delta GEF$	
i. $\overline{BC} \cong \overline{EF}$	i. Given
ii. $\angle B \cong \angle GEF$	ii. Construction
iii. $\overline{BA} \cong \overline{GE}$	iii. Construction
$\therefore$ $\Delta ABC \cong \Delta GEF$	S.A.S. postulate.
$\therefore$ $\overline{AC} \cong \overline{GF}$ and $\angle A \cong \angle G$	By the congruencw of triangles.
But $\overline{DF} \cong \overline{AC}$	Given
$\therefore$ $\overline{GF} \cong \overline{DF}$	Transitive property.
$\therefore$ In ΔDEG, $m\angle 1 = m\angle 3$	Opposite sides congurent $\overline{EG} \cong \overline{BA} \cong \overline{ED}$

Similarly, in ΔGFE, $m\angle 2 = m\angle 4$	$\overline{DF} \cong \overline{GF}$
$\therefore m\angle 1 + m\angle 2 = m\angle 3 + m\angle 4$	Addition property of equation
or $m\angle D = m\angle G$	$m\angle 1 + m\angle 2 = m\angle D$ $m\angle 3 + m\angle 4 = m\angle G$
But $m\angle G = m\angle A$	Proved above
$\therefore m\angle A = m\angle D$	Transitive property
In $\Delta ABC \leftrightarrow \Delta DEF$	
i. $\overline{AB} \cong \overline{DE}$	i. Given
ii. $\angle A \cong \angle D$	ii. Proved above
iii. $\overline{AC} \cong \overline{DF}$	iii. Given
$\therefore \Delta ABC \cong \Delta DEF$	S.A.S Postulate

Q.E.D

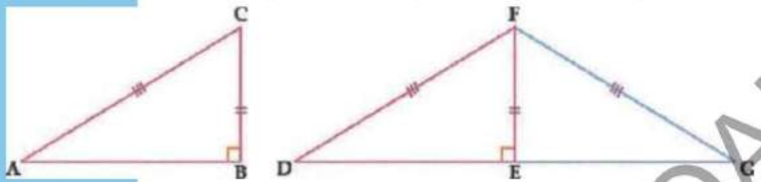
Corollary: The angles of an equilateral triangle are also equal in measurement.

Exercise 9.3

1. ABC is an isosceles triangle. D is the mid-point of base $\overline{BC}$. Prove that $\overline{AD}$ bisects $\angle A$ and $\overline{AD} \perp \overline{BC}$.
2. ABC and DBC are two isosceles triangles on the same side of a common base $\overline{BC}$. Prove that $\overleftrightarrow{AD}$ is the right bisector of $\overline{BC}$.
3. PQRS is a square. X,Y and Z are the mid-points of $\overline{PQ}$, $\overline{QR}$ and $\overline{RS}$ respectively. Prove that $\Delta PXY \cong \Delta SZY$.

Theorem 9.1.4

If in the correspondence of two right-angled triangles, the hypotenuse and one side of one are congruent to the hypotenuse and the corresponding side of the other, then the triangles are congruent (H.S≅H.S).



Given: In correspondence

$\Delta ABC \leftrightarrow \Delta DEF$

$\angle B \cong \angle E$ (rt $\angle$s) $\overline{AC} \cong \overline{DF}$ (Hyp) and $\overline{BC} \cong \overline{EF}$

To Prove: $\Delta ABC \cong \Delta DEF$

Construction: Produce $\overline{DE}$ to point G such that $\overline{EG} \cong \overline{AB}$. Then join F and G.

Proof:

Statements	Reasons
$m\angle DEF + m\angle GEF = 180^o$	Supplement postulate
But $m\angle DEF = 90^o$	Given
$\therefore$ $m\angle GEF = 90^o$	$\therefore$ $180^o - 90^o = 90^o$
In $\Delta GEF \leftrightarrow \Delta ABC$	
i. $\overline{GE} \cong \overline{AB}$	Construction
ii. $\angle GEF \cong \angle ABC$	Each is rt angle
iii. $\overline{EF} \cong \overline{BC}$	Given
$\therefore$ $\Delta GEF \cong \Delta ABC$	S.A.S. ≅ S.A.S.
$\therefore$ $\overline{FG} \cong \overline{FG}$ and $\angle G \cong \angle A$	By the congruence of Δs.
$\therefore$ $\overline{FG} \cong \overline{DF}$	$\therefore$ $\overline{AC} \cong \overline{DF}$ (Given)
In ΔDFG, $\angle D \cong \angle G$	Opposite sides congruents
$\therefore$ $\angle D \cong \angle A$	Each is congruent to ΔG
In $\Delta ABC \leftrightarrow \Delta DEF$	
i. $\angle A \cong \angle D$	i. Proved
ii. $\angle ABC \cong \angle DEF$	ii. rt Δs
iii. $\overline{AC} \cong \overline{DF}$	iii. Given
$\therefore$ $\Delta ABC \cong \Delta DEF$	A.A.S ≅ A.A.S

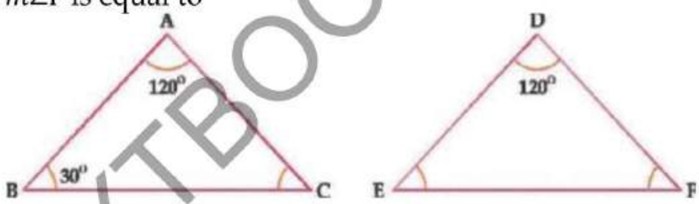
Q.E.D

Exercise 9.4

1. Prove that:
   The perpendiculars from the vertices of the base to opposite sides of an isoceles triangle are congruent.
2. Prove that, if the bisector of an angle of a triangle bisects its opposite side, then the triangle will be an isosceles triangle.
3. Prove that, in an equilateral triangle any two median are congruent.
4. Prove that the median bisecting the base of an isosceles triangle bisect the vertical angle and is perpendicular to the base.
5. Prove that if three altitudes of a triangle are congruent, then the triangle is equilatera

Review Exercise 9

1. If $\Delta ABC \cong \Delta DEF$, $m\angle F$ is equal to
   A. 90°
   B. 60°
   C. 30°
   D. 20°



2. Identify true and false statement in the following:
   (i) The sum of the measure of all angles in an quadrilateral is 360°.
   (ii) The sum of the measure of all angles in a triangle is 270°.
   (iii) In an equilateral triangle, angles are of the same measurement.
   (iv) There are two right angles in a traiangle.
   (iv) In an isosceles triangle, corresponding angles and correspoinding sides are equal in measure.
3. Fill in the blanks to make the sentences true sentences:
   (i) In $\Delta ABC \leftrightarrow \Delta DEF$, then $\overline{AC}$ corresponds to ____________.
   (ii) In $\Delta KLM \leftrightarrow \Delta PQR$, then $\angle MKL$ corresponds to ____________
   (iii) In an isosceles triangle, the base angle are ____________.
   (iv) If the mesure of each of the angles of a triangle is 60°, then the triangle is____________.
   (v) In a right-angled triangle, side opposite to right angle is called ____________.
   (vi) The sum of the measures of acute angle of a right triangle is____________.

4. Encircle the corresponding letters a,b,c or d for correct answer:

(i) Which of the following is not a sufficent condition for congurence of two triangles?

(a) A.S.A ≅A.S.A (b) H.S.H ≅ H.S.H

(c) S.A.A≅ S.A.A (d) A.A.A≅ A.A.A

(ii) In ΔABC, if ∠A ≅ ∠B, then the bisector of _____ angle divides the triangle into congruent triangles:

(a) ∠A (b) ∠B

(c) ∠C (d) any one of its angles.

(iii) The diagonal of ____ does not divide it into two congruent triangles:

(a) Rectangle (b) Trapezium

(c) Parallogram (d) Square

(iv) How many acute angles are there in an acute angled triangle?

(a) 1 (b) 2

(c) 3 (d) not more than 2.

Summary

In this unit we stated and proved the following theorem:

- In any correspondence of two traingles, if one side and any two angles of one traingles are congruent ti the corresponding side and angles of the other, the two taringles are congruent. (A.S.A. ≅ A.S.A)
- If two angles of a traingles Are congruent, then the side opposite to them are also congruent.
- In the correspondence of two traingles, if three sides of two traingles are congruent to the corresponding three sides of other, then the two traingles are congruent (S.S.S ≅ S.S.S).
- If in the correspondence of the two right-angled traingles the hypotenuse and one side of one traingles are congruent to the hypotenuse and the corresponding side of other, then the traingles are congruent. (H.S ≅ H.S).
- Two traingles are said ti be congruent, if there exists a correspondence beteew them such that all the corresponding sides and traingles are congruent.(S.S.S).