Unit 4

FACTORIZATION

Student Learning Outcomes (SLOs)

After completing this unit, students will be able to:

- Recall factorization of expressions of the following types.
  - $ka + kb + kc$ (Common factors in all the terms)
  - $ac + ad + bc + bd$(Common factors in pairs of terms)
  - $a^2 \pm 2ab + b^2$ (Perfect squares)
  - $a^2 - b^2$(Difference of two squares)
  - $(a^2 \pm 2ab + b^2) - c^2$
  - $(\sqrt{a})^2 - (\sqrt{b})^2$
- Factorize the expressions of the following types.

  Type I: $a^4 + a^2b^2 + b^4$ and $a^4 + b^4$

  Type II: $x^2 + px + q$

  Type III: $ax^2 + bx + c$

  Type IV: $(ax^2 + bx + c)(ax^2 + bx + d) + k$

  $(x + a)(x + b)(x + c)(x + d) + k$

  $(x + a)(x + b)(x + c)(x + d) + kx^2$

  Type V: $a^3 + 3a^2b + 3ab^2 + b^3$ and $a^3 - 3a^2b + 3ab^2 - b^3$

  Type VI: $a^3 \pm b^3$
- State and prove remainder theorem and explain through examples.
- Find remainder (without dividing) when a polynomial is divided by a linear polynomial.
- Define zero of a polynomial.
- State and prove factor theorem.
- Describe the method of synthetic division.
- Use synthetic division to:
  - Find quotient and remainder when a given polynomial is divided by a linear polynomial.
  - Find the value(s) of unknown(s) if the zeros of the polynomial are given.
  - Find the value(s) of unknown(s) if the factors of the polynomial are given.
- Use factor theorem to factorize a cubic polynomial.

Introduction

we will study about factorization which has an important role in mathematics. It helps us to reduce the complicated expression into simple expressions.

4.1 Factorization

Let, $p(x)$,$q(x)$ and $r(x)$ are three polynomials such that, $p(x)\times q(x) = r(x)$, here, the resulting polynomial $r(x)$ is the product of $p(x)$ and $q(x)$,and the polynomials $p(x)$ and $q(x)$ are called the **factors of $r(x)$.**

There are some examples of factors of the polynomials are given below.

(i) $6x^2y^3=(2\times 3)(x\times x)(y\times y\times y)$

(ii) $ax + aby + abcz = a(x + by + bcz)$

(iii) $5x + 15xy= 5x(1+3y)$

(iv) $x - y= \left(\sqrt{x}\right)^2 - \left(\sqrt{y}\right)^2 = \left(\sqrt{x}-\sqrt{y}\right)\left(\sqrt{x}+\sqrt{y}\right)$

4.1.1 Recall Factorization of Expressions of the Following Types

(i) $ka + kb + kc$ (Common factors in all the terms)

(ii) $ac + ad + bc + bd$ (Common factors in pairs of terms)

(iii) $a^2 \pm 2ab + b^2$ (Perfect squares)

(iv) $a^2 - b^2$ (Difference of two squares)

(v) $a^2 \pm 2ab + b^2 - c^2$

(vi) $\left(\sqrt{a}\right)^2 - \left(\sqrt{b}\right)^2$

(i) Factors of the type: $ka + kb + kc$

Let us see the following examples

Example 01 Factorize: $10a + 15b - 20c$

Solution: $10a + 15b - 20c$

$= 5(2a + 3b - 4c)$ (5 as a common from the expression)

Example 02 Find the factors of $\frac{4}{9}-\frac{8}{12}x-\frac{16}{15}xy$

Solution: $\frac{4}{9}-\frac{8}{12}x-\frac{16}{15}xy$

$=\frac{4}{3\cdot 3}-\frac{4\cdot 2}{3\cdot 4}x-\frac{4\cdot 4}{3\cdot 5}xy$

$=\frac{4}{3}\left(\frac{1}{3}-\frac{2}{4}x-\frac{4}{5}xy\right)$ (Taking $\frac{4}{3}$ as a common from the expression)

(ii) Factors of the type: $ac + ad + bc + bd$

See the following examples

Example 01 Find the factors of $3a - ac - 3c + c^2$

Solution: $3a - ac - 3c + c^2$

$= a(3 - c) - c(3 - c)$

$= (3 - c)(a - c)$

Example 02 $9y^2z + 3xyz - 9xy^2 - 3x^2y$

Solution: $9y^2z + 3xyz - 9xy^2 - 3x^2y$

$= 9y^2z + 3xyz - 9xy^2 - 3x^2y$

$= 3yz(3y + x) - 3xy(3y + x)$

$= (3y + x)(3yz - 3xy)$

$= (3y + x) \times 3y(z - x)$

$= 3y(x + 3y)(z - x)$ are the required factors.

(iii) Factors of the type: $a^2 \pm 2ab + b^2$

As we know that,

$$a^2 + 2ab + b^2 = (a)^2 + 2(a)(b) + (b)^2 = (a + b)^2$$

and $$a^2 - 2ab + b^2 = (a)^2 - 2(a)(b) + (b)^2 = (a - b)^2$$

Let us see the following examples

Example 01 Factorize $16a^2 + 40ab + 25b^2$

Solution: $16a^2 + 40ab + 25b^2$

$= (4a)^2 + 2(4a)(5b) + (5b)^2$

$= (4a + 5b)^2$ $[\because a^2 + 2ab + b^2 = (a + b)^2]$

Example 02 Factorize $4p^2 - 28pq + 49q^2$

Solution: $4p^2 - 28pq + 49q^2$

$= (2p)^2 - 2(2p)(7q) + (7q)^2$

$= (2p - 7q)^2$ $[\because a^2 - 2ab + b^2 = (a - b)^2]$

(iv) Factors of the type: $a^2 - b^2$

Let us see following examples

Example 01 Factorize $4x^2 - 1$

Solution: $4x^2 - 1$

$= (2x)^2 - (1)^2$

$= (2x - 1)(2x + 1)$ $[\because a^2 - b^2 = (a - b)(a + b)]$

Example 02 Factorize $96y^2 - 6z^2$

Solution: $96y^2 - 6z^2$

$= 6(16y^2 - z^2)$

$= 6[(4y)^2 - z^2]$

$= 6(4y - z)(4y + z)$ $[\because a^2 - b^2 = (a - b)(a + b)]$

Example 03 Factorize $9r^4 - (6s - t^2)^2$

Solution: $9r^4 - (6s - t^2)^2$

$= (3r^2)^2 - (6s - t^2)^2$

$= [3r^2 - (6s - t^2)][3r^2 + (6s - t^2)]$, $[\because a^2 - b^2 = (a - b)(a + b)]$

$= (3r^2 - 6s + t^2)(3r^2 + 6s - t^2)$

(v) Factors of the type: $(a^2 \pm 2ab + b^2) - c^2$

Let us see following examples.

Example 01 Factorize $x^2 + 4xy^2 + 4y^4 - 4z^2$

Solution: $x^2 + 4xy^2 + 4y^4 - 4z^2$

$= \{(x)^2 + 2(x)(2y^2) + (2y^2)^2\} - (2z)^2$

$= (x + 2y^2)^2 - (2z)^2$

$= \{(x + 2y^2) + 2z\}\{(x + 2y^2) - 2z\}$ $[\because a^2 - b^2 = (a - b)(a + b)]$

$= (x + 2y^2 + 2z)(x + 2y^2 - 2z)$

Example 02 $9p^2 - 6pq + q^2 - 9r^2$

Solution: $9p^2 - 6pq + q^2 - 9r^2$

$= (3p)^2 - 2(3p)(q) + q^2 - (3r)^2$

$= (3p - q)^2 - (3r)^2$

$= (3p - q + 3r)(3p - q - 3r)$ $[\because a^2 - b^2 = (a - b)(a + b)]$

(vi) Factors of the type: $\left(\sqrt{a}\right)^2 - \left(\sqrt{b}\right)^2$

Example 01 Factorize $\left(\sqrt{xy}\right)^2 - \left(\sqrt{z}\right)^2$

Solution: $\left(\sqrt{xy}\right)^2 - \left(\sqrt{z}\right)^2$

$= \left(\sqrt{xy} - \sqrt{z}\right)\left(\sqrt{xy} + \sqrt{z}\right)$ $[\because a^2 - b^2 = (a - b)(a + b)]$

Exercise 4.1

1. Factorize the following:

(i) $4x + 16y + 24z$ (ii) $x^2 + 3x^2y + 4x^2y^2z$

(iii) $3pqr + 6pqt + 3pqs$ (iv) $9qr(s^2 + t^2) + 18q^2r^2(s^2 + t^2)$

(v) $\frac{z^2x}{16} - \frac{x^2z^2}{8} + \frac{x^2z^3}{12}$ (vi) $a(x-y) - a^2b(x-y) + a^2b^2(x-y)$

2. Factorize the following:

(i) $7x + xz + 7z + z^2$ (ii) $9a^2b + 18ab^2 - 6ac - 12bc$

(iii) $6t - 12p + 4tq - 8pq$ (iv) $r^2 + 9rs - 7rs - 63s^2$

(v) $\frac{y^2}{4} - \frac{4^2z}{4} - \frac{z^2t}{9} + \frac{z^2tz}{9}$ (vi) $\frac{10xy}{11} + \frac{5xz}{11} - \frac{14yz}{11} - \frac{7xy}{11}$

3. Find the factors of:

(i) $4a^2 + 12ab + 9b^2$ (ii) $36x^4 + 12x^2 + 1$

(iii) $x^2 + 1 + \frac{1}{4x^2}$ (iv) $81y^2 + 144yz + 64z^2$

(v) $625 + 50a^2b + a^4b^2$ (vi) $a^2 + 0.4a + 0.04$

4. Factorize:

(i) $b^4 - 4b^2c^2 + 4c^4$ (ii) $\frac{9}{4}x^4 - 2 + \frac{4}{9x^4}$

(iii) $2a^3b^3 - 16a^2b^4 + 32ab^5$ (iv) $9(p+q)^2 - 6(p+q)r^2 + r^4$

(v) $x^2y^2 - 0.1xy + 0.0025$ (vi) $(a-b)^2 - 18(a-b) + 81$

5. Factorize:

(i) $4a^2 - 9b^2$ (ii) $16x^2 - 25y^2$ (iii) $100x^2z^2 - y^4$

(iv) $\frac{1}{100}x^4 - 100y^4$ (v) $\frac{64}{81}f^2 - \frac{81}{64}g^4$ (vi) $\frac{x^4}{121} - 121y^2$

6. Find the factors of:

(i) $(2x + z)^2 - (2x - z)^2$ (ii) $(4a - 9b)^2 - (2a + 5b)^2$

(iii) $169x^4 - (3t + 4)^2$ (iv) $(9x^2 - 4y^2)^2 - (4x^2 - y^2)^2$

(v) $\left(a^2 + 2 + \frac{1}{a^2}\right) - \left(b^2 - 2 + \frac{1}{b^2}\right)$ (vi) $9x^2 + \frac{1}{9x^2} - 4y^2 - \frac{1}{4y^2} + 4$

7. Find the factors of:

(i) $(x^2 + 2xy + y^2) - 9z^4$ (ii) $(4a^2 + 8ab^2 + 4b^4) - 9c^2$

(iii) $16d^4 - (c^4 - 2c^2d + d^2)$ (iv) $4(x^2 + 2xy^2 + y^4) - 9y^6$

(v) $x^2 - y^2 - 4x - 2y + 3$ (vi) $4x^2 - y^2 - 2y - 1$

8. **Factorize:**

(i) $\left(\sqrt{ab}\right)^2 - \left(\sqrt{c}\right)^2$ (ii) $\left(\sqrt{4x}\right)^2 - \left(\sqrt{9y}\right)^2$

(iii) $\left(\sqrt{yz}\right)^2 - \left(\frac{1}{\sqrt{yz}}\right)^2$ (iv) $xzt - \frac{1}{t}$

4.1.2 Factorize the expression of following types.

Type I: $a^4 + a^2b^2 + b^4$ or $a^4 + 4b^4$

This type includes those algebraic expressions which are neither perfect squares nor in the form of the difference of two squares. Factorization of this type is explain in the following examples.

Example 01 Factorize: $a^4 + a^2b^2 + b^4$.

Solution: $a^4 + a^2b^2 + b^4$

$= (a^4 + b^4) + a^2b^2$ (Rearrange the terms)

$= (a^4 + 2a^2b^2 + b^4) - 2a^2b^2 + a^2b^2$ [by adding and subtracting $2a^2b^2$]

$= \{(a^2)^2 + 2(a^2)(b^2) + (b^2)^2\} - a^2b^2$

$= (a^2 + b^2)^2 - (ab)^2$

$= \{(a^2 + b^2) - ab\}\{(a^2 + b^2) + ab\}$ [$\because a^2 - b^2 = (a - b)(a + b)$]

$= (a^2 - ab + b^2)(a^2 + ab + b^2)$ are the required factors.

Example 02 Factorize: $a^4 + 4b^4$

Solution: $a^4 + 4b^4$

$= (a^2)^2 + (2b^2)^2$

$= (a^2)^2 + (2b^2)^2 + 2(a^2)(2b^2) - 2(a^2)(2b^2)$ [by adding and subtracting $2(a^2)(2b^2)$]

$= \{(a^2)^2 + 2(a^2)(2b^2) + (2b^2)^2\} - 4a^2b^2$

$= (a^2 + 2b^2)^2 - (2ab)^2$

$= \{(a^2 + 2b^2) - 2ab\}\{(a^2 + 2b^2) + 2ab\}$ [$\because a^2 - b^2 = (a - b)(a + b)$]

$= (a^2 - 2ab + 2b^2)(a^2 + 2ab + 2b^2)$ are the required factors.

Example 03 Factorize: x^8+x^4+1

Solution: x^8+x^4+1

$= (x^8+1)+x^4$ [by adding and subtracting $2(x^4)(1)$]

$= \{(x^4)^2+(1)^2+2(x^4)(1)\}-2(x^4)(1)+x^4$

$= (x^4+1)^2-x^4$

$= (x^4+1)^2-(x^2)^2$

$= \{(x^4+1)-x^2\}\{(x^4+1)+x^2\}$ $[\because a^2-b^2=(a-b)(a+b)]$

$= \{(x^4+x^2+1)\}(x^4-x^2+1)$

$= \{(x^2+1)^2-2x^2+x^2\}(x^4-x^2+1)$

$= \{(x^2+1)^2-x^2\}(x^4-x^2+1)$

$= (x^2+x+1)(x^2-x+1)(x^4-x^2+1)$ are the required factors.

Type II: x^2+px+q

This type of expression can be factorized by breaking the middle term process.

Example 01 Factorize $y^2+7y+12$

Solution: $y^2+7y+12$

$= y^2+3y+4y+12$

$= y(y+3)+4(y+3)$

$= (y+3)(y+4)$

Example 02 Factorize $x^2+13xy-30y^2$

Solution: $x^2+13xy-30y^2$

$= x^2+15xy-2xy-30y^2$

$= x(x+15)-2y(x+15)$

$= (x+15)(x-2y)$

Type III: $ax^2+bx+c,\ a\neq 0.$

To factorize the expression ax^2+bx+c, $a\neq 0$, the following steps are needed:

(i) Find the product ac, where a is coefficient of x^2 and c is constant.

(ii) Find two numbers x_1 and x_2 such that $x_1+x_2=b$ and $x_1x_2=ac$.

To explain this method the following example is helpful.

Example 01 Factorize $10x^2-19xy+6y^2$

Solution: $10x^2-19xy+6y^2$

$= 10x^2 - 15xy - 4xy + 6y^2$

$= 5x(2x - 3y) - 2y(2x - 3y)$

$= (2x - 3y)(5x - 2y)$

Example 02 Factorize $4x^2+12x+5$

Solution: $4x^2+12x+5$

$= 4x^2 + 10x + 2x + 5$

$= 2x(2x + 5) + 1(2x + 5)$

$= (2x + 5)(2x + 1)$

Exercise 4.2

1. Factorize the following:

(i) $a^4+a^2x^2+x^4$ (ii) b^4+b^2+1

(iii) $a^8+a^4x^4+x^8$ (iv) z^8+z^4+1

2. Factorize:

(i) $42x^2-8x-2$ (ii) $21z^2-4z-1$

(iii) $9y^2+21yz^4-8y^2$ (iv) $24a^2-18a+27$

3. Factorize:

(i) x^4+4y^2 (ii) $36x^4z^4+9y^4$

(iii) $4t^4+625$ (iv) $4t^4+1$

4. Resolve into factors:

(i) $x^2+3x-10$ (ii) $a^2b^2-3ab-10$

(iii) $y^2+7y-98$ (iv) $x^2y^2z^2+2xyz-24$

5. Resolve into factors:

(i) $121x^4+11x^2+2$ (ii) $42z^4+50z^2+8$

(iii) $4x^2+12x+5$ (iv) $3x^2-38xy-13y^2$

6. Resolve into factors:

(i) $81x^4 + 36x^2y^2 + 16y^4$ (ii) $x^4 + x^2 + 25$

(iii) $y^4 - 7y^2 - 8$ (iv) $16a^4 - 97a^2b^2 + 81b^4$

Type IV: $(ax^2 + bx + c)(ax^2 + bx + d)+k$

$(x+a)(x+b)(x+c)(x+d)+k$

$(x+a)(x+b)(x+c)(x+d)+kx^2$

We shall explain the procedure of factorizing of these types expressions with the help of following examples.

Example 01 Factorize: $(x^2+5x+4)(x^2+5x+6) - 120$

Solution: $(x^2+5x+4)(x^2+5x+6) - 120$

Let $x^2+5x = t$, then we have

$(t+4)(t+6) - 120$

$= t^2+10t+24 - 120$

$= t^2+10t - 96$

$= t^2 - 6t + 16t - 96$ (by factorizing)

$= t(t-6) + 16(t-6)$

$= (t-6)(t+16)$

$= (x^2+5x-6)(x^2+5x+16)$ $\because t = x^2+5x$

$= (x^2 - x + 6x - 6)(x^2+5x+16)$

$= [x(x-1) + 6(x-1](x^2+5x+16)$

$= (x-1)(x+6)(x^2+5x+16)$

Example 02 Factorize: $(x+1)(x+2)(x+3)(x+4) - 15$

Solution: $(x+1)(x+2)(x+3)(x+4) - 15$

Here 1+4= 2+3 =5

$(x+1)(x+4)(x+2)(x+3) - 15$ by arranging the factors

$= (x+1)(x+4)(x+2)(x+3) - 15$

$= (x^2+5x+4)(x^2+5x+6) - 15$

$= (t+4)(t+6) - 15$ where $t = x^2+5x$

$= t^2+10t+24-15$

$= t^2+10t+9$

$= (t+1)(t+9)$

$= (x^2+5x+1)(x^2+5x+9) \quad \because t = x^2+5x$

Example 03 Factorize: $(x+2)(x-2)(x-3)(x+3)+(-2x^2)$

Solution: $(x+2)(x-2)(x-3)(x+3)+(-2x^2)$

$= (x+2)(x-2)(x-3)(x+3)+(-2x^2)$

$= (x^2-2^2)(x^2-3^2)-2x^2 \qquad [\because (a+b)(a-b)=a^2-b^2]$

$= (x^2-4)(x^2-9)-2x^2$

$= x^4 - 9x^2 - 4x^4 + 36 - 2x^2$

$= x^4 - 15x^2 + 36$

$= x^4 - 3x^2 - 12x^2 + 36$

$= x^2(x^2-3) - 12(x^2-3)$

$= (x^2-3)(x^2-12)$

$= [(x)^2 - (\sqrt{3})^2][(x)^2 - (2\sqrt{3})^2]$

$= (x-\sqrt{3})(x+\sqrt{3})(x-2\sqrt{3})(x+2\sqrt{3})$

Exercise 4.3

1. Factorize the following:

(i) $(x^2-4x-5)(x^2-4x-12)-144$
(ii) $(x^2+5x+6)(x^2+5x+4)-3$
(iii) $(x^2-2x+3)(x^2-2x+4)-42$
(iv) $(x^2-8x+4)(x^2-8x-4)+15$
(v) $(x^2+9x-1)(x^2+9x+5)-7$
(vi) $(x^2-5x+4)(x^2-5x+6)-120$

2. Factorize:

(i) $(x+1)(x+2)(x+3)(x+4)-48$
(ii) $(x+2)(x+3)(x+4)(x+5)+24$
(iii) $(x-1)(x-2)(x-3)(x-4)-99$
(iv) $(x-3)(x-5)(x-7)(x-9)+15$
(v) $(x-1)(x-2)(x-3)(x-4)-224$
(vi) $(x-2)(x-3)(x-4)(x-5)-255$

3. Find the factors of:

(i) $(x-2)(x-3)(x+2)(x+3)-2x^2$

(ii) $(x-1)(x+1)(x+3)(x-3)-3x^2-23$

(iii) $(x-1)(x+1)(x-3)(x+3)+4x^2$

(iv) $(x-2)(x+2)(x-4)(x+4)-14x^2$

(v) $(x+5)(x+2)(x-5)(x-2)+4x^2$

(vi) $(x^2-x-12)(x^2-x-12)-x^2$

Type V: $a^3+3a^2b+3ab^2+b^3$ **and** $a^3-3a^2b+3ab^2-b^3$

As we know that

$$a^3+3a^2b+3ab^2+b^3=(a+b)^3$$

and $$a^3-3a^2b+3ab^2-b^3=(a-b)^3.$$

The following examples will help us to understand the factorization of the types mentioned above.

Example 01 Factorize (i) $8x^3+12x^2y+6xy^2+y^3$ (ii) $64x^3-12x^2+\frac{3x}{4}-\frac{1}{64}$

Solution (i): $8x^3+12x^2y+6xy^2+y^3$

$=(2x)^3+3(2x)^2(y)+3(2x)(y)^2+(y)^3$ $[\because (a^3+3a^2b+3ab^2+b^3=(a+b)^3]$

$=(2x+y)^3$.

Solution (ii): $64x^3-12x^2+\frac{3x}{4}-\frac{1}{64}$

$=(4x)^3-3(4x)^2(\frac{1}{4})+3(4x)\left(\frac{1}{4}\right)^2-\left(\frac{1}{4}\right)^3$ $[\because (a^3-3a^2b+3ab^2-b^3=(a-b)^3]$

$=\left(4x-\frac{1}{4}\right)^3$

Exercise 4.4

1. Factorize the following:

(i) $b^3+3b^2c+3bc^2+c^3$ (ii) $8x^3+12x^2y+6xy^2+y^3$

(iii) $64x^3+12x^2+\frac{3x}{4}+\frac{1}{64}$ (iv) $8x^3+36x^2+54x+27$

(v) $\frac{1}{27}+\frac{1}{3}y^2+y^4+y^6$ (vi) $\frac{8}{27}x^3+2x^2y+\frac{9}{2}xy+\frac{27}{8}y^3$

(vii) $\frac{64}{27}+\frac{16}{3}x^2+4x+x^3$ (viii) $\frac{z^3}{8}+\frac{z^2y}{4}+\frac{zy^2}{6}+\frac{y^3}{27}$

2. Find the factors of:

(i) $d^3-6d^2c+12dc^2-8c^3$ (ii) $x^6+\frac{16}{3}x^4-4x^2-\frac{64}{27}$

(iii) $\frac{x^3}{125}+\frac{3}{25}x^2y-\frac{3}{36}xy^2-y^3$ (iv) $125z^3-75z^2y^2+15zy^4-y^6$

(v) $\frac{z^3}{27}-\frac{2z^2y}{x}+zy^2-216y^3$ (vi) $\frac{b^6}{27}-\frac{b^4c^2}{6}+\frac{b^2c^4}{4}-\frac{c^6}{8}$

(vii) $216+\frac{9}{2}z^2-54z-\frac{z^3}{8}$ (viii) $\frac{8}{27}x^3-2x^2y+\frac{9}{2}xy-\frac{27}{8}y^3$

Type VI: $a^3 \pm b^3$

As we know that,

$$a^3+b^3=(a+b)(a^2-ab+b^2),$$

and $$a^3-b^3=(a-b)(a^2+ab+b^2).$$

The following example will help us to understand the factorization of above mentioned type.

Example 01 Factorize $8x^3+27$

Solution: $8x^3+27$

$= (2x)^3+(3)^3$

$= (2x+3)[(2x)^2-(2x)(3)+(3)^2]$ $[\because a^3+b^3=(a+b)(a^2-ab+b^2)]$

$= (2x+3)(4x^2-6x+9)$

Therefore: $8x^3+27=(2x+3)(4x^2-6x+9)$

Example 02 factors of $108x^4 - 256xz^3$.

Solution: $108x^4 - 256xz^3$

$= 4x(27x^3 - 64z^3)$

$= 4x[(3x)^3 - (4z)^3$ $\qquad [\because a^3 - b^3 = (a-b)(a^2 + ab + b^2)]$

$= 4x(3x - 4z)[(3x)^2 + (3x)(4z) + (4z)^2]$

$= 4x(3x - 4z)(9x^2 + 12xz + 16z^2)$

Therefore: $108x^3 - 256xz^3 = 4x(3x - 4z)[(9x^2 + 12xz + 16z^2)$.

Exercise 4.5

1. **Factorize the following:**

(i) $x^3 + 8y^3$ (ii) $a^{11} + a^2b^9$ (iii) $a^6 + 1$ (iv) $a^3b^3 + 512$

(v) $a^3b^3 + 27b^6$ (vi) $\dfrac{x^3}{125} + \dfrac{125}{x^3}$ (vii) $x^9 + x^3y^6z^9$ (viii) $\dfrac{x^6}{27} + \dfrac{8}{x^3}$

2. **Find the factors of:**

(i) $x^3 - 8y^3$ (ii) $x^9 - 8y^9$ (iii) $1000 - \dfrac{x^3y^3}{125}$ (iv) $a^6 - b^6$

(v) $\dfrac{x^6}{64} - \dfrac{64}{x^{12}}$ (vi) $x^{12} - y^{12}$ (vii) $\dfrac{27}{x^3} - 8y^6$ (viii) $8x^6 - \dfrac{1}{729}$

4.2 Remainder and Factor Theorems

Remainder and factor theorems are usually used to find the factors of the polynomial expressions of third and higher degrees of the polynomials.

4.2.1 State and prove Remainder Theorem and Explain through examples

Statement:

When a polynomial $p(x)$ of degree $n \geq 1$ is divided by $(x-a)$ The Remainder R is found by $R = p(a)$.

We can write $p(x)$ as $p(x) = q(x)(x-a) + R$, (Which is called division algorithm) where R is a constant (Reminder) and the degree of $q(x)$ is one less than that of $p(x)$.

Proof:

By division algorithm

$p(x) = q(x)(x-a)+R,$

Let $x=a$ then,

$p(a) = q(a)\times(a-a)+R,$

$p(a) = q(a)\times 0+R$

$\Rightarrow \quad p(a) = R =$ remainder.

4.2.2 Find remainder (without dividing) when a polynomial is divided by a linear polynomial.

The following examples help us to use of remainder theorem.

Example 01 Find the remainder when x^2-3x+4 is divided by $x-2$

Solution: Let $p(x)=x^2-3x+4$

Here a=2 by remainder theorem

$p(2) = (2)^2 - 3(2) + 4$

$= 4 - 6 + 4 = 8 - 6 = 2$

$p(2) = R = 2$

Thus, the reminder is 2.

Example 02 Find the value of k, if the polynomial x^3+kx^2+3x-4 leaves a remainder -2 when divided by $x+2$.

Solution:

Here $p(x) = x^3+kx^2+3x-4$

$\therefore \quad p(-2) = (-2)^3 + k(-2)^2 + 3(-2) - 4$

$\Rightarrow \quad -2 = 4k - 18$ Reminder $= -2$

$\Rightarrow \quad 4k = -2 + 18$

$\Rightarrow \quad 4k = 16$

$\Rightarrow \quad k = 4,$

Thus, the value of k is 4.

4.2.3 Zero of a polynomial.

Let $p(x) = a_0 + a_1x + a_2x^2 + \ldots\ldots + a_nx^n$ be a polynomial with real coefficient. By putting $x=a$ in the polynomial $p(x)$, if the value of $p(x)$ becomes zero i.e. $p(a)=0$. Then 'a' is called the zero of polynomial $p(x)$.

Example $p(x) = x+7$ then -7 is a zero of polynomial as $p(-7)=-7+7=0$.

4.2.4 State and prove factor theorem.

Statement:

The linear polynomial $x–a$ is a factor of the polynomial $p(x)$ if, $p(a) = 0$.

Proof:

Let $q(x)$ be quotient and R be the remainder when a polynomial $p(x)$ is divided by $x–a$

Then by division algorithm, we have

$p(x)=(x-a)q(x)+R.$

By the Remainder theorem,

$R = p(a),$

Thus, $\boxed{p(x) = q(x)(x-a)+p(a)}$

If $p(a)=0$, then, $p(x)=q(x)(x-a)$. Thus, $(x- a)$ is one of the factors of $p(x)$.

The following examples will help us to use of factor theorem.

Example 01 Determine whether $x + 2$ is a factor of $x^3 + \frac{9}{2}x^2 + 3x - 4$ or not.

Solution: Let $p(x) = x^3 + \frac{9x^2}{2} + 3x - 4$

$$\therefore R = p(-2) = (-2)^3 + \frac{9}{2}(-2)^2 + 3(-2) - 4$$

$$= -8 + 18 - 6 - 4$$

$$= -18 + 18 = 0 \Rightarrow R = 0,$$

Remainder = 0,

$\therefore$ $x+2$ is factor of $p(x)$

Example 02 Determine whether $x + 3$ factor of is $x^3 - x^2 - 8x + 12$

Solution:

Let $p(x) = x^3 - x^2 - 8x + 12$

$R = p(-3) = (-3)^3 - (-3)^2 - 8(-3) + 12$

$= -27 - 9 + 24 + 12$

$R = p(-3) = -36 + 36 = 0$

$x+3$ is factor of $x^3 - x^2 - 8x + 12$

Exercise 4.6

1. Find the remainder by using the remainder theorem when

(i) $x^3 - 6x^2 + 11x - 8$ is divided by $(x-1)$

(ii) $x^3 + 6x^2 + 11x + 8$ is divided by $(x+1)$

(iii) $x^3 - x^2 - 26 + 40$ is divided by $(x-2)$

(iv) $x^3 - 3x^2 + 4x - 14$ is divided by $(x+2)$

(v) $(2y-1)^3 + 6(3+4y) - 9$ is divided by $(2y+1)$

(vi) $4y^3 - 4y + 3$ is divided by $(2y-1)$

(vii) $(2y+1)^3 - 6(3-4y) - 10$ is divided by $(2y-1)$

(viii) $x^4 + x^2y^2 + y^4$ is divided by $(x-y)$.

4. Find the value of m, if $p(y) = my^3 + 4y^2 + 3y - 4$ and $q(y) = y^3 - 4y + m$ leaves the same remainder when divided by $(y-3)$.

5. If the polynomial $4x^3 - 7x^2 + 6x - 3k$ is exactly divisible by $(x + 2)$, find the value of k.

6. Find the value of r, if $(y+2)$ is a factor of the polynomial $3y^2 - 4ry - 4r^2$.

4.3 Synthetic Division

Synthetic division is a method to divide a polynomial by a linear polynomial.

4.3.1 Describe the method of synthetic division.

The method of synthetic division is described with the help of following example.

Example 1. Divide the polynomial $p(x) = x^3 - 3x^2 + 5x + 7$ *by* $(x-1)$ using synthetic division

Solution: Here $x - 1 = 0 \Rightarrow x = 1$, (1 is a multiplier).

Write the coefficient of the polynomial.

Thus,

1	1	−3	5	7	(Row 1)
		1	−2	3	(Row 2)
	1	−2	3	10= R	(Row 3)

Description

Step I. In Row 1, write the coefficients and constant of the polynomial $p(x)$ in the descending order.

Step II. Write the first coefficient in Row 3, below its position in Row 1.

Step III. Write the product of 1 (the multiplier) and the coefficient (1) in the Row 3 beneath the 2nd coefficient in Row 2, and add, putting the sum below them in the Row 3 and so on.

Thus, $q(x)=x^2-2x+3$ and $p(1) = R = 10$.

Note that:	Degree of $q(x)$=[Degree of $p(x)$]−1= 3−1=2 and the last element of the row3, is the remainder.

4.3.2 Use of Synthetic division to:

(a) Find quotient and remainder when a given polynomial is divided by a linear polynomial.

(b) Find the value(s) of unknown(s) if zeros of the polynomial are given.

(c) Find the value(s) of unknown(s) if the factors of the polynomial are given.

Example 01 Find quotient and remainder when,
$p(x) = x^4-12x^3+50x^2-84x+49$ is divided by a linear polynomial $x-5$

Solution: Given that $p(x)=x^4-12x^3+50x^2-84x+49$,
and linear polynomial $x-a=x-5$, i.e., $a=5$ is the multiplier = 5
$p(5)=R=?$ and $q(x)=?$

To find quotient and the remainder, we will use synthetic division illustrate as under.

5	1	−12	50	−84	49	(Row 1)
		5	−35	75	−45	(Row 2)
	1	−7	15	−9	4 = R	(Row 3)

Thus, $q(x)=x^3-7x^2+15x-9$ and $R=4$ are the required quotient and remainder respectively.

Note: $R\neq 0$, therefore $(x-5)$ is not the factor of given polynomial $p(x)$.

Example 02 For what value of m, 1 is a zero of the polynomial $p(x)= x^3-mx^2+x-1$?

Solution: Given that,

$p(x)= x^3-mx^2+x-1$

Here, the multiplier $a= 1$.

By synthetic division method we have,

1	1	$-m$	1	-1	(Row 1)
		1	$1-m$	$2-m$	(Row 2)
	1	$1-m$	$2-m$	$1-m = R,$	(Row 3)

Since 1 is the zero of p(x) $\therefore$ $x - 1$ is factor

Here, $R= 0$

$\Rightarrow \quad 1-m=0$

$\Rightarrow \quad m=1$

Thus, for zero of the polynomial $p(x)$ m must be equal to 1.

Exercise 4.7

1. By using synthetic division method to divide the following polynomials and also find their quotient and remainder.

(i) $p(x)=x^3-x^2+x-1$ by $x-1$ (ii) $p(x)=x^3-x^2-x-1$ by $x+1$

(iii) $p(x)=x^3-6x^2+11x-6$ by $x+2$ (iv) $p(x)=x^3+6x^2-11x-6$ by $x-2$

(v) $p(x)=x^4-x^3+x^2-x-1$ by $x+2$ (vi) $p(x)=x^4+x^3-x^2+x-1$ by $x-1$

(vii) $p(x)=x^5+x^3-2x^2-3$ by $x+3$ (viii) $p(x)=x^5-x^4+x^3-3x^2+6x-6$ by $x-3$

(ix) $p(x)=2x^4-2x^3+100x^2-168x+95$ by $x-2$

(x) $p(x)=6x^4-72x^3+300x^2-564x+270$ by $x-5$

2. For what value of k, -2 is zero of the polynomial $p(x)=x^3+x^2-14x-k$

3. For what value of m, $(x-2)$ be factor of $x^3+mx^2-7x-10$

4. For what value of m, $(x+2)$ is factor of $4x^3-7x^2+6x-3m$

5. For what value of k, -1 is a zero of the polynomial, $P(x)=2x^3-4mx^2+x-1$

4.4 Factorization of Cubic Polynomial

We have already studied the method of solving linear and quadratic polynomials. Now we will find the factors of cubic polynomials using factor theorem.

4.4.1 Use factor theorem to factorize a cubic polynomial

To factorize a cubic polynomial by factor theorem, it is necessary that one of the factor or more of the zeros of the polynomial is (are) known.

Let us see the following examples:

Example 01 Find the factors of $x^3 - 6x^2 + 11x - 6$

Solution: Let $p(x) = x^3 - 6x^2 + 11x - 6$

The factors of 6 are $\pm1, \pm2, \pm3, \pm6$

$p(1) = (1)^3 - 6(1)^2 + 11(1) - 6$

$p(1) = 1 - 6 + 11 - 6$

$p(1) = 0$

Hence, $x-1$ is a factor of $p(x)$

By synthetic division

1	1	−6	11	−6	(Row 1)
		1	−5	6	(Row 2)
	1	−5	6	0	(Row 3)

$p(x) = (x-1)(x^2 - 5x + 6)$

$= (x-1)\{x^2 - 2x - 3x + 6\}$

$= (x-1)\{x(x-2) - 3(x-2)\}$

$p(x) = (x-1)(x-2)(x-3)$

Example 02 Find the factor of $x^3 - 4x^2 + x + 6$

Solution: The factor of 6 are $\pm1, \pm2, \pm3, \pm6$

If $x-1$ is a factor of $p(x)$

Then,

$p(1) = (1)^3 - 4(1)^2 + 1 + 6$

$= 1 - 4 + 1 + 6$

$= 4 \neq 0$

Hence, x–1 is not factor of $p(x)$.
If x+1 is a factor of $p(x)$
Then,

$$p(-1)=(-1)^3-4(-1)^2+1(-1)+6$$
$$=-1-4-1+6$$
$$=0$$

Hence, x–1 is factor of $p(x)$.
By synthetic division

–1	1	–4	1	6	(Row 1)
		–1	5	–6	(Row 2)
	1	–5	6	0	(Row 3)

$$p(x)=(x+1)(x^2-5x+6)$$
$$=(x+1)\{x^2-2x-3x+6\}$$
$$=(x+1)\{x(x-2)-3(x-2)\}$$
$$p(x)=(x+1)(x-2)(x-3)$$

Exercise 4.8

Find the factors by using factor theorem

1. x^3-x^2+x-1 **2.** x^3-x^2-x-1 **3.** $x^3-6x^2+11x-6$
4. $x^3+5x^2-4x-20$ **5.** x^3+x^2-x-1 **6.** $x^3-2x^2+9x-18$
7. $6x^3+7x^2-x-2$ **8.** $x^3+8x^2+19x+12$ **9.** $2x^3+9x^2+10x+3$
10. $x^3+7x^2+14x+8$ **11.** $x^3+9x^2+26x+24$ **12.** $x^3+12x^2+44x+48$

Review Exercise 4

1. True and false questions

Read the following sentences carefully and encircle 'T' in case of true and 'F' in case of false statement.

(i) $x^2+x-6=(x+3)(x-2)$ T/F
(ii) $a^3+27=(a+3)(a^2-3a+9)$ T/F
(iii) $b^3-8=(b-2)(b^2+2b+4)$ T/F

(iv) $a^4 - b^4 = (a-b)(a+b)(a+b)^2$ T/F

(v) $a^6+b^6=(a^3+b^3)(a^3-b^3)$ T/F

(vi) $a^5+b^5=(a+b)^5$ T/F

2. Complete the following sentences

(i) $16x^2-y^4=(4x-y^2)$ ____________

(ii) $x^3-64y^3=(x-4y)$ ____________

(iii) $x^2+5x+6=(x+2)$ ____________

(iv) $x^2+y^2=(x-y)^2$ ____________

(v) $a^3+27b^3=(a+3b)$ ____________

3. Tick (✓) the correct answers

(i) Factors of $a^2+2a-24$ are:

(a) $a+4,\ a-6$ (b) $a-4,\ a+6$

(c) $a+3,\ a-8$ (d) $a+8,\ a-3$

(ii) Factors of $a^2+2ab+b^2-c^2$ is:

(a) $(a-b+c)(x-b-c)$ (b) $(a+b+c)(a-b-c)$

(c) $(a+b+c)(a+b-c)$ (d) $(a+b+c)(a-b-c)$

(iii) Factors of x^3+y^3 is:

(a) $(x-y)(x^2+xy+y^2)$ (b) $(x+y)(x^2+xy+y^2)$

(c) $(x+y)(x^2+xy-y^2)$ (d) $(x+y)(x^2-xy+y^2)$

(iv) Factors of y^3-27z^3 are:

(a) $y-3z,\ y^2+3yz+9z^2$ (b) $y-3z,\ y+3z+9z^2$

(c) $y-3z,\ y^2-3yz+9z^2$ (d) $y+3z,\ y^2-3yz+9z^2$

(v) In simplified form $\frac{1}{x+y}+\frac{y}{x^2-y^2}=$

(a) $\frac{y+1}{x^2-y^2}$ (b) $\frac{x}{x^2-y^2}$

(c) $\frac{y}{x^2-y^2}$ (d) $\frac{y-1}{x^2-y^2}$

(vi) Find m, so that x^2+4x+m is a complete square

(a) 8 (b) –8

(c) 4 (d) –4

Summary

- Factorization is the process in which we express the given polynomial (expression) as a product of two or more expressions.
- We will learn and resolve into factors of the following types of formulas:
  - (i) $ka + kb + kc = k(a + b + c)$
  - (ii) $\underline{ac + ad} + \underline{bc + bd} = a(c + d) + b(c + d) = (a + b)(c + d)$
  - (iii) $a^2 + 2ab + b^2 = (a)^2 + 2(a)(b) + (b)^2 = (a + b)^2$
  - (iv) $a^2 - 2ab + b^2 = (a)^2 - 2(a)(b) + (b)^2 = (a - b)^2$
  - (v) $a^2 - b^2 = (a-b)(a + b)$
- Remainder and factor theorems are two important theorems; these are used to find the factors of such type of polynomials which cannot be solved by the given formulas:
  - (i) $a^3 + 3a^2b + 3ab^2 + b^3 = (a+b)^3$
  - (ii) $a^3 - 3a^2b + 3ab^2 - b^3 = (a-b)^3$
  - (iii) $a^3 + b^3 = (a+b)(a^2 - ab + b^2)$
  - (iv) $a^3 - b^3 = (a-b)(a^2 + ab + b^2)$.
- **Zeros of the Polynomial**
  If specific number $x=a$ is substituted for the variable 'x' in a polynomial $p(x)$ such that, the value of $p(a)$ is zero, then 'a' is called a zero of the Polynomial $p(x)$.
- **Factor theorem** can also be stated as:
  The linear polynomial $(x - a)$ is a factor of the polynomial $p(x)$ if, $p(a)=0$
- **Description of synthetic division method**

  Step I. Write in Row 1, the coefficients of $p(x)$ in the descending powers of x.

  Step II. Write the first coefficient in Row below its position in Row 1.

  Step III. Write the product of 2 (multiplier) and this coefficient in the Row 2 beneath the 2nd coefficient in Row 1, and added, putting the sum below them in the Row 3 and so on.