GENERAL MATHEMATICS

For Class

9

Punjab Curriculum & Textbook Board, Lahore

UNIT 1

PERCENTAGE, RATIO AND PROPORTION

- Percentage
- Ratio
- Proportion
- Compound Proportion

After completion of this unit, the students will be able to:

- Know percentage as a fraction with denominator of 100.
- Convert:
  - A percentage to a fraction by expressing it as a fraction with denominator 100.
  - A fraction to a percentage by multiplying it with 100%.
  - A percentage to a decimal and vice versa.
- Solve real life problems involving percentage.
- Know:
  - A ratio as a relation, which one quantity bears to another quantity of the same kind with regard to their magnitudes.
  - That, of the two quantities forming a ratio, the first one is called antecedent and the second one consequent.
  - That a ratio has no units.
  - The importance of the order in which the ratio is expressed.
- Find the ratio when a number is increased (decreased) to become another number (e.g., in what ratio must 40 be decreased to become 24?)
- Solve real life problems involving ratios.
- Know that an equality of two ratios $\left(\frac{a}{b}=\frac{c}{d}\right)$ constitutes a proportion, that is, $a:b::c:d$, where a,d are known as extremes and b,c are called the means.
- Find proportion (direct and inverse).
- Solve real life problems involving direct and inverse proportion.
- Know the concept of compound proportion.
- Solve real life problems involving compound proportion.

1.1 PERCENTAGE

The word "percent" is a short form of the Latin word "percentum". Percent means out of hundred or per hundred. The symbol for percentage is "%".

1.1.1 Percentage as a Fraction with Denominator 100

40% *means* 40 *out of* 100, *i.e* $\frac{40}{100}$

60% *means* 60 *out of* 100, *i.e* $\frac{60}{100}$

85% *mean* 85 *out of* 100, *i.e* $\frac{85}{100}$

1.1.2 Conversion of a Percentage to a Fraction by Expressing it as a Fraction with Denominator 100

We can write the percentage as a fraction with denominator *100* as in the following examples:

30% *means* 30 *out of* 100, *i.e.* $30\% = \frac{30}{100}$

55% *means* 55 *out of* 100, *i.e* $55\% = \frac{55}{100}$

EXAMPLE

Express 70%, $22\frac{1}{2}\%$ and $45\frac{1}{2}\%$ as a fraction in their lowest form.

SOLUTION:

$$70\% = \frac{70}{100} = \frac{7}{10}$$

$$22\frac{1}{2}\% = \frac{45}{2\times 100}$$

$$= \frac{9}{2\times 20}$$

$$= \frac{9}{40}$$

$$45\frac{1}{2}\% = \frac{91}{2\times 100}$$

$$= \frac{91}{200}$$

Convert a Fraction to Percentage by Multiplying it with 100%

To convert a fraction to a percentage by multiplying it with *100%*, let us consider the following examples.

EXAMPLE

Express the following fractions as a percentage.

(i) $\frac{7}{20}$ (ii) $\frac{9}{20}$ (iii) $\frac{7}{5}$ (iv) $\frac{1}{3}$

SOLUTION:

(i) $\frac{7}{20} = \frac{7}{20}\times 100\%$
$= 7\times 5\%$
$= 35\%$

(ii) $\frac{9}{20} = \frac{9}{20}\times 100\%$
$= 9\times 5\%$
$= 45\%$

(iii) $\frac{7}{5} = \frac{7}{5}\times 100\%$
$= 7\times 20\%$
$= 140\%$

(iv) $\frac{1}{3} = \frac{1}{3}\times 100\%$
$= \frac{100}{3}\%$
$= 33\frac{1}{3}\%$

Convert a Percentage to a Decimal and Decimal to a Percentage

To convert a percentage to a decimal, we look at the following examples.

EXAMPLE-1

Express (i) 54% (ii) $16\frac{1}{2}\%$ (iii) $27\frac{1}{3}\%$ as decimals.

SOLUTION: (i) $54\% = \frac{54}{100}$

$= 0.54$

(ii) $16\frac{1}{2}\% = \frac{16.5}{100}$

$= \frac{165}{1000}$

$= 0.165$

(iii) $27\frac{1}{3}\% = \frac{82}{3\times 100}$

$= \frac{27.3}{100}$

$= \frac{273}{1000}$

$= 0.273$

71% of the Earth's surface is water.
Write the percentage of land. ☐

EXAMPLE-2

Aslam scored 35 marks out of 50 in English, 60 out of 75 in Urdu and 72 out of 75 in Pakistan Studies. In which subject did he perform best?

SOLUTION: Marks percentage in English $= \frac{35}{50}\times 100$

$= 35\times 2$

$= 70\%$

Marks percentage in Urdu $= \frac{60}{75} \times 100$

$= 20 \times 4$

$= 80\%$

Marks percentage in Pakistan Studies $= \frac{72}{75} \times 100$

$= 24 \times 4$

$= 96\%$

So, Aslam performed best in Pakistan Studies.

EXAMPLE-3

Express (i) 0.7 (ii) 0.13 (iii) 1.26 as percentage.

SOLUTION: (i) $0.7 = 0.7 \times 100\%$

$= \frac{7}{10} \times 100\%$

$= 7 \times 10\%$

$= 70\%$

(ii) $0.13 = 0.13 \times 100\%$

$= \frac{13}{100} \times 100\%$

$= 13\%$

(iii) $1.26 = 1.26 \times 100\%$

$= \frac{126}{100} \times 100\%$

$= 126\%$

Convert a Percentage to a Decimal and Decimal to a Percentage

To convert a percentage to a decimal, we look at the following examples.

EXAMPLE-1

Express (i) 54% (ii) $16\frac{1}{2}\%$ (iii) $27\frac{1}{3}\%$ as decimals.

SOLUTION: (i) $54\% = \frac{54}{100}$

$$= 0.54$$

(ii) $16\frac{1}{2}\% = \frac{16.5}{100}$

$$= \frac{165}{1000}$$

$$= 0.165$$

(iii) $27\frac{1}{3}\% = \frac{82}{3\times 100}$

$$= \frac{27.3}{100}$$

$$= \frac{273}{1000}$$

$$= 0.273$$

> 71% of the Earth's surface is water.
> Write the percentage of land. ☐

EXAMPLE-2

Aslam scored 35 marks out of 50 in English, 60 out of 75 in Urdu and 72 out of 75 in Pakistan Studies. In which subject did he perform best?

SOLUTION: Marks percentage in English $= \frac{35}{50}\times 100$

$$= 35\times 2$$

$$= 70\%$$

$$\text{Marks percentage in Urdu} = \frac{60}{75} \times 100 = 20 \times 4 = 80\%$$

$$\text{Marks percentage in Pakistan Studies} = \frac{72}{75} \times 100 = 24 \times 4 = 96\%$$

So, Aslam performed best in Pakistan Studies.

EXAMPLE-3

Express (i) 0.7 (ii) 0.13 (iii) 1.26 as percentage.

SOLUTION: (i) 0.7

$$0.7 = 0.7 \times 100\% = \frac{7}{10} \times 100\% = 7 \times 10\% = 70\%$$

(ii) 0.13

$$0.13 = 0.13 \times 100\% = \frac{13}{100} \times 100\% = 13\%$$

(iii) 1.26

$$1.26 = 1.26 \times 100\% = \frac{126}{100} \times 100\% = 126\%$$

EXERCISE – 1.1

1. Express the following percentages as fractions in their lowest form.

   (i) 95% (ii) 65% (iii) 75% (iv) 25% (v) 56% (vi) 48%

   (vii) 8% (viii) $33\frac{1}{2}\%$ (ix) $37\frac{1}{2}\%$ (x) $87\frac{1}{2}\%$ (xi) $5\frac{1}{4}\%$ (xii) $42\frac{1}{2}\%$

2. Express the following fractions as percentages, giving your answer correct to *1* decimal place, where necessary.

   (i) $\frac{3}{4}$ (ii) $\frac{3}{5}$ (iii) $\frac{4}{25}$ (iv) $\frac{13}{20}$ (v) $\frac{31}{25}$ (vi) $\frac{21}{40}$

   (vii) $\frac{23}{60}$ (viii) $\frac{8}{3}$ (ix) $\frac{8}{5}$ (x) $\frac{7}{8}$ (xi) $\frac{5}{8}$ (xii) $\frac{3}{8}$

3. Express the following percentages as decimals, giving your answer correct to *3* places of decimal.

   (i) 47% (ii) 58% (iii) 92% (iv) 8% (v) 12% (vi) 120%

   (vii) 180% (viii) 145% (ix) $5\frac{1}{2}\%$ (x) $5\frac{1}{3}\%$ (xi) $48\frac{2}{3}\%$ (xii) $58\frac{1}{3}\%$

4. Express the following decimals as percentages.

   (i) 0.5 (ii) 0.9 (iii) 1.25 (iv) 1.39 (v) 1.72 (vi) 0.22

   (vii) 2.64 (viii) 3.41 (ix) 0.845 (x) 1.78 (xi) 1.58 (xii) 0.065

5. Complete the following table.

	Fraction	Percentage	Decimal
I.	$\frac{3}{4}$	75%	0.75
II.	$\frac{4}{5}$		0.80
III.		40%	
IV.			0.62
V.		44%	

1.1.3 Real Life Problems Involving Percentage

Consider the following examples from real life involving percentage.

EXAMPLE-1

If there are 800 cars in a car parking and 80% of them are Pakistan made, find the number of Pakistani cars.

SOLUTION: *Total number of cars in a car parking = 800*

$$80\% = \frac{80}{100}$$

Number of Pakistani cars $= \frac{80}{100} \times 800$

$= 640$

EXAMPLE-2

If $\frac{4}{5}$ of the students in a school have been away for a holiday. How many in every hundred have been on holiday?

SOLUTION: $\frac{4}{5} = \frac{4}{5} \times 100\% = 4 \times 20\% = 80\%$

$\frac{4}{5}$ *of the students have been away means 80 in every hundred have been on holiday.*

EXAMPLE-3

If 56% of the houses of a colony have a car, what percentage of houses do not have cars ?

SOLUTION:

Number of houses have a car $= 56\ \%$

Houses do not have a car $= (100 - 56)\ \%$

$= 44\ \%$

Thus 44 % houses do not have a car.

If we are given one percentage out of two, we can deduce the other.

EXERCISE – 1.2

1. If *45%* of the students in a school are girls. What is the percentage of the boys?

2. If *82%* of the houses have a television, what is the percentage of the houses which do not have?

3. A hockey team won *62%* of their matches and *26 %* of them were ended in a draw. What is the percentage of the matches they lost?

4. An aeroplane carries 400 passengers, *52%* of the passengers were Pakistani, *17%* were Chinese, *12%* were from Iran and the rest were British.

   (i) How many people of each nationality were on the plane ?

   (ii) What is the percentage of the British ?

5. Amna scored *46* out of *50* in a Maths test, *64* out of *75* in a Chemistry test and *72* out of *80* in a Physics test. In which subject did she perform best ?

6. A table costs a carpenter *Rs. 720* to make. He sells it for *Rs. 920*. What is the percentage of profit he earned?

7. If *84 %* of a book consists of *420* pages. Find total number of pages in the book.

8. Out of his total income Hamza spends *20 %* on house rent and *70 %* of the rest on household expenditure. If he saves *Rs. 1800*, what is his total income ?

9. Raheel's income is *25 %* more than that of Rauf. What percent is Rauf's income less than Raheel's ?

1.2 RATIO

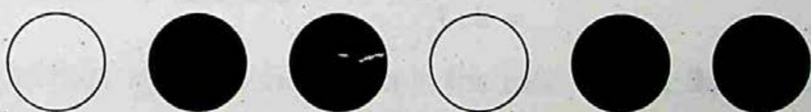
In our previous classes we have learnt about ratios and solved problems involving ratios. Let us recall that here.

In our daily life we are always in need of comparing values or magnitude of objects. For example if there are *6* eggs in one basket and *24* eggs in other basket. The comparison of number of eggs in both the baskets, leads to the concept of ratio.

So ratio is a comparison of like quantities measured in like units. The symbol for ratio is ':'

1.2.1(i) Ratio as a Relation

Here are six balls.



There are 2 white balls out of 6 balls.

The fraction of the balls that are white is $\frac{2}{6} = \frac{1}{3}$.

There are *2* white balls and *4* black balls. The ratio of white balls to black balls is $2 : 4$.

We simplify ratios like fractions i.e., $2 : 4 = 1 : 2$.
The ratio $1 : 2$ tells us that there is one red ball to every *2* blue balls. The ratio compares the number of red balls with the number of blue balls.

If a and b represent two quantities, where b is not zero, ratio of a to b is written as $a:b$ or in terms of fraction $\frac{a}{b}$.

If one quantity has the magnitude *2* and the other has the magnitude *3*, then the ratio of the two quantities is *2* to *3*, written as $2 : 3$, or $\frac{2}{3}$.

If the two quantities in comparison are not in the same units, then to find their ratio, we convert them first in the same units, e.g. if the lengths of the two scales are *50cm* and *3m* then the ratio is:

$$\left.\begin{array}{l} 50:300 \ \text{(in centimeters)} \\ 1:6 \end{array}\right\} \quad (100cm = 1m)$$

A ratio $a:b$ is said to be in its simplest form when a and b are integers with no common factor *(other than 1)*.

EXAMPLE-1

Simplify the ratio 8:12 in the simplest form.

SOLUTION: $8:12 = \frac{8}{4}:\frac{12}{4}$

$= 2:3$

EXAMPLE-2

Simplify the ratio 24:12 in the simplest form.

SOLUTION: $24:12 = \frac{24}{12}:\frac{12}{12}$

$= 2:1$

1.2.1.(ii) Antecedent and Consequent

In a ratio $a:b$, 'a' is called the antecedent and 'b' is called the consequent, e.g. in ratio $2:5$, antecedent is 2 and consequent is 5.

1.2.1.(iii) Ratio has no Units

Let us consider a jug and a glass with *1500ml* and *200ml* juice. Comparison of the volumes of the juice in two objects.

$$\frac{\text{Volume of the juice in the glass}}{\text{Volume of the juice in the jug}} = \frac{200\,ml}{1500\,ml} = \frac{2}{15} \text{ or } 2:15$$

We can compare these quantities because the numerator and denominator, are in the same units, therefore, ratio $2:15$ has no units.

1.2.1.(iv) The Order of a Ratio

If the magnitudes of the two quantities are denoted by 'a' and 'b' then ratio from 'a' to 'b' is $a:b$.

We cannot write this ratio as $b:a$, because, $a:b \neq b:a$ *since* $\frac{a}{b} \neq \frac{b}{a}$.

Therefore, in a ratio the order of quantities must be maintained;
e.g. $2:5$ and $5:2$ are different ratios because $2:5 \neq 5:2$ or $\frac{2}{5} \neq \frac{5}{2}$.

1.2.2 Ratio when a Number is Increased or Decreased

If the number of Mathematics books in a school library are increased from 75 to 95, then the ratio of the previous number of books to the present number of books is $= 75 : 95$

$= 15 : 19$

i.e. the number of books increased in the ratio of $15 : 19$.

EXAMPLE-1

A student spends Rs.70 everyday, but on Sunday, he spends Rs.20 only. Find the ratio of number of rupees spent on Sunday to everyday.

SOLUTION: *The ratio of number of rupees spent on Sunday comparing to the other days is* $= 20 : 70$

$= 2 : 7$

EXAMPLE-2

Increase 40 books in the ratio 5 : 4. What would be the number of increased books ?

SOLUTION: *Number of books* $= 40$

Given ratio is $= 5:4 = \frac{5}{4}$

Number of books increased $= 40 \times \frac{5}{4}$

$= 10 \times 5$

$= 50$

The number of books increased is 50.

EXAMPLE-3

In what ratio $60\,m^2$ be decreased to $24\,m^2$.

SOLUTION: *Required ratio = new magnitude : old magnitude*

$$= 24 : 60$$

$$= \frac{24}{60}$$

$$= \frac{2}{5}$$

$$= 2 : 5$$

EXERCISE – 1.3

1. Find the ratio of first quantity to the second in its lowest form.

(i) Rs. 24 , Rs. 6 (ii) 20 kg , 5 kg

(iii) 20cm , 80cm (iv) 5m , 5m

(v) 1500 km , 1200 km (vi) Rs. 150 , Rs. 275

2. Express each of the following ratios in its simplest form.

(i) $\frac{2}{3} : \frac{3}{5}$ (ii) $\frac{4}{5} : \frac{3}{4}$ (iii) $\frac{5}{6} : \frac{7}{10}$

(iv) $\frac{13}{40} : \frac{3}{20}$ (v) $\frac{2}{3} : \frac{1}{6}$ (vi) $\frac{4}{10} : 20$

(vii) $\frac{15}{10} : 2$ (viii) $\frac{12}{10} : \frac{28}{10}$ (ix) $\frac{2}{5} : \frac{1}{3}$

3. In a city *126* medical students traveled by:

Rikshaw	Taxi	Bus	Car
14	9	75	28

Find ratio of the students who used.

(i) Rikshaw to taxi
(ii) Taxi to bus
(iii) Taxi to car.

4. In a school library, there are *75* books on Mathematics, *115* on English, *85* on Chemistry and *60* on Physics. Find ratio of the following:

(i) Mathematics books to English books.
(ii) English books to Chemistry books.
(iii) English books to Physics books.
(iv) Physics books to Chemistry books
(v) Physics books to Mathematics books.
(vi) Chemistry books to Mathematics books.

1.2.3 Real Life Problems Involving Ratio

Some of the problems relating to our daily life are given in the following:

EXAMPLE-1
There are 1029 students in a school, 504 out of them are girls, what is the ratio of boys to the number of girls?

SOLUTION: *Total number of students* $= 1029$

Number of girls $= 504$

Number of boys $= 1029 - 504$

$= 525$

Required ratio $= 525 : 504$

EXAMPLE-2

A rectangle has length of $6cm$ and width of $4cm$.

A second rectangle has a length of $9cm$ and a width of $2cm$.

Find the ratios of: *(i) their lengths* *(ii) their widths* *(iii) their perimeters* *(iv) their areas*

SOLUTION: Given length of the first rectangle $= 6cm$

Width of the first rectangle $= 4cm$

Area of the first rectangle $= 6 \times 4$

$= 24\ cm^2$

Perimeter of the first rectangle $= 2 \times (6 + 4)$

$= 20cm$

Length of the second rectangle $= 9cm$

Width of the second rectangle $= 2cm$

Area of the second rectangle $= 9 \times 2$

$= 18cm^2$

Perimeter of the second rectangle $= 2 \times (9 + 2)$

$= 22cm$

(i) Ratio of length of the first rectangle to the length of the second $= 6 : 9$

$= 2 : 3$

$= \frac{2}{3}$

(ii) Ratio of width of the first rectangle to the width of the second $= 4 : 2$

$= 2 : 1$

(iii) Ratio of perimeter of the first rectangle to the second is $= 20 : 22$

$= 10 : 11$

(iv) Ratio of area of the first rectangle to the second is $= 24 : 18$

$= 4 : 3$

EXAMPLE-3

A couple has 6 grandsons and 4 granddaughters.
Find the ratios of:

(i) the number of grandsons to that of granddaughters.
(ii) the number of granddaughters to that of grandsons.

SOLUTION:

Number of grandsons $= 6$

Number of granddaughters $= 4$

grandsons : granddaughters $= 6:4$

$= 3:2$

granddaughters : grandsons $= 4:6$

$= 2:3$

EXAMPLE-4

Find the ratio of:

(i) 8 rupees each to 72 rupees per dozen.
(ii) 36 rupees per dozen to 6 rupees each.

SOLUTION: (i) 72 rupees per dozen means

$$\frac{72}{12} = 6 \text{ rupees each.}$$

Therefore, ratio of 8 rupees each to 72 rupees per dozen is same as the ratio of 8 rupees to 6 rupees, i.e.

$$8 : 6 = 4 : 3$$

Required ratio is: $4 : 3$

(ii) 36 rupees per dozen means

$$\frac{36}{12} = 3 \text{ rupees each.}$$

Therefore, ratio of 36 rupees per dozen to 6 rupees each is same as the ratio of 3 rupees to 6 rupees, i.e.

$$3 : 6 = 1 : 2$$

Required ratio is: $1 : 2$

EXAMPLE-5

If $a : b = 2 : 3$, find the ratio of $6a : 5b$.

SOLUTION: Given $a : b = 2 : 3$ then

$$6a : 5b = 6\times2 : 5\times3$$
$$= 12 : 15$$
$$= \frac{12}{3} : \frac{15}{3}$$
$$= 4 : 5$$

Thus $6a : 5b = 4 : 5$

EXERCISE – 1.4

1. Find the ratio of 6 rupees each to 72 rupees per dozen.
2. Find the ratio of *Rs*. 160 per meter to *Rs*. 150 per meter.
3. Find the ratio of *Rs*. 72 for 24 to rupees 4 each.
4. A square '*A*' has side 2*cm* and a square '*B*' has side 6*cm*. Find the ratio of:
   - *(i)* The length of the side of the square *A* to the length of the side of the square *B*.
   - *(ii)* The perimeter of the square *A* to the perimeter of the square *B*.
   - *(iii)* The area of the square '*A*' to the area of the square '*B*'.
5. If $a : b = 2 : 3$, find the ratio $6a : 2b$.
6. A triangle has sides of lengths 3*cm* , 4*cm* and 6*cm*. Find the ratio of the lengths of the sides to one another.
7. Two angles in a triangle are 54° and 72°. Find the ratio of the third angle to the sum of the first two.
8. Ali's father earns a salary of *Rs*. 40,000 in a month, while his father's monthly expenditures are *Rs*. 35,000. Find the ratio of his father's:
   - *(i)* Income to expenditure
   - *(ii)* Expenditure to savings
   - *(iii)* Income to savings

9. A square A has side $6cm$ and square B has side $8cm$.
Find the ratio of:
   - *(i)* The length of the side of a square A to the length of the side of the square B.
   - *(ii)* The area of square A to the area of square B.

10. A family has 12 pets of which 6 are cats, 2 are dogs and the rest are birds. Find the ratio of the number of:
   - *(i)* birds to dogs
   - *(ii)* birds to pets

1.3 PROPORTION

The equality of two ratios is known as proportion. The symbol for proportion is "::" or "=".

1.3.1 Extremes and Means

If $a : b = c : d$, then the proportion is $a : b :: c : d$. We read it as ratio a is to b is as ratio c is to d.

a,b,c and d are called the terms of the proportion.

The first and fourth terms, i.e "a" and "d" are called the extremes, while the second and third terms "b" and "c" are called the means of the proportion.

The product of means is equal to the product of extremes, i.e.

Extremes

$$a : b :: c : d$$

Means

$$ad = bc$$

EXAMPLE

Find the unknown term in the proportion $x : 3 :: 60 : 15$

SOLUTION:

$$\text{Product of extremes} = ad = x \times 15$$
$$= 15x$$
$$\text{Product of means} = cb = 3 \times 60$$
$$= 180$$
$$\text{Product of extremes} = \text{Product of means}$$

Therefore, $15x = 180$

$$x = \frac{180}{15}$$
$$x = 12$$

Thus the unknown term is 12

1.3.2 Proportion (Direct and Inverse)

Direct Proportion

The relationship between two ratios in which increase in one quantity causes a proportional increase in the other quantity or decrease in one quantity causes a decrease in the other quantity is called "direct proportion".

Inverse Proportion

The relationship between two ratios in which increase in one quantity causes a proportional decrease in the other quantity or a decrease in the one quantity causes a proportional increase in the other quantity is an inverse proportion.

1.3.3 Real Life Problems

EXAMPLE-1

The price of 20 pens is Rs.2000. What will be the price of 40 such pens ?

SOLUTION: *Let "x" be the price of 40 pens, Then*

Pens	Price
20 ↓	2000 ↓
40	x

Therefore $20 : 40 :: 2000 : x$ or $20 : 40 = 2000 : x$

$$\frac{20}{40} = \frac{2000}{x}$$

$$\frac{1}{2} = \frac{2000}{x}$$

$$x \times 1 = 2 \times 2000$$

$$x = 4000$$

Thus price of 40 pens will be Rs.4000.

EXAMPLE-2

The price of 80 shirts is Rs.22000. What would be the price of 30 such shirts ?

SOLUTION: *Let x be the required price, then*

Shirts	Price
80 ↓	22000 ↓
30	x

Therefore $80 : 30 :: 22000 : x$

$$\frac{80}{30} = \frac{22000}{x}$$

$$80x = 22000 \times 30$$

$$x = \frac{22000 \times 30}{80}$$

$$x = 8250$$

Thus price of 30 shirts is Rs.8250.

EXAMPLE-3

In a school hostel of 300 students, a food stock for 30 days was present. Later on 50 students left the hostel. For how many days the same food will be sufficient for the remaining students?

SOLUTION:

Total number of students in the hostel = 300

Number of students left the hostel = 50

Remaining students = 300 − 50

= 250

Let x be the required number of days. As the number of students decrease the number of days will increase in the proportion.

Students	Days
↓ 300	30 ↑
250	x

Therefore, $300 : 250 :: x : 30$

$$\frac{300}{250} = \frac{x}{30}$$

$$250 \times x = 300 \times 30$$

$$x = \frac{300 \times 30}{250}$$

$$x = 36$$

Thus the stock of food shall last for 36 days.

EXAMPLE-4

6 persons can do a job in 12 days. If 2 more persons are employed, how many days will they take to complete the job?

SOLUTION: Number of persons = 6

Number of persons increased = 2

Total number of persons = 6 + 2

= 8

The job was completed = 12 days.

Let the job shall be completed in "x days", when the number of persons are increased.

Persons	Days
↓ 6	12 ↑
8	x

$$6 : 8 :: x : 12$$

$$\frac{6}{8} = \frac{x}{12}$$

$$8x = 12 \times 6$$

$$x = \frac{72}{8} \Rightarrow x = 9$$

Thus the number of days required is 9.

EXAMPLE-5

An army formation of 900 men has a food stock for 30 days. Later on 150 army men leave the formation. For how many days the same food will be sufficient for remaining army men ?

SOLUTION: Total men = 900

Men left = 150

Remaining men = 750

Let x be the required number of days.

Men	Days
↓ 900	30 ↑
750	x

Therefore $900 : 750 :: x : 30$

$$\frac{900}{750} = \frac{x}{30} \quad \Rightarrow \quad x = \frac{90 \times 30}{75}$$

$$= 6 \times 6$$

$$= 36 \text{ days}$$

Thus the number of days required is 36.

1.4 COMPOUND PROPORTION

The relationship between two or more proportions is known as compound proportion. For details we see the following examples.

EXAMPLE-1

A shopkeeper plans to produce 200 articles with the help of 5 persons working 8 hours daily. How many articles can be made by 8 persons if they work 6 hours daily?

SOLUTION: *Let x be the required number of articles, we have.*

Persons	Working hours	Articles
5	8	200
8	6	x

If number of persons increases, then the number of articles also increases } *direct proportion*

If the working hours decreases, then the number of articles decreases } *direct proportion*

Persons	Working hours daily	Articles
↑ 5	↑ 8	200 ↑
8	6	x

$$\frac{x}{200} = \frac{6}{8} \times \frac{8}{5}$$

$$x = \frac{6}{8} \times \frac{8}{5} \times 200$$

$$= 240$$

Required number of articles is 240.

EXAMPLE-2

Rs. 4000 are sufficient for a family of 4 members for 40 days. For how many days Rs.15,000 will be sufficient for a family of 5 members?

SOLUTION: *Let x be the required number of days. Placing the given quantities in table, we have*

Rupees	*Members*	*Days*
4000	4	40
15,000	5	x

Since the number of days is required. So we compare the last column of above table with the first two columns as follows.

If rupees increase, then the number of days increases } → *Direct Proportion*

If members increase, then the number of days decreases } → *Indirect Proportion*

Rupees	*Members*	*Days*
↑ 4000	↓ 4	↑ 40
15000	5	x

Using arrow signs, we have

$$\frac{x}{40} = \frac{4}{5} \times \frac{15000}{4000}$$

$$\Rightarrow x = \frac{4}{5} \times \frac{15000}{4000} \times 40$$

$$\Rightarrow x = 120$$

This shows that the food will be sufficient for 120 days.

EXERCISE – 1.5

1- Find the value of x in the proportion $20 : 50 :: 8 : x$?

2- The price of *15* suits is *Rs.6750*. How many such suits can be purchased by an amount of *Rs.4050*?

3- A motorcycle covers *90km* in *2* liters of petrol. In how many liters of petrol will it cover *225km*?

4- A certain journey by train takes *5* hours at the speed of *45 km/h*. What will be the speed of the train to complete the same journey in *3* hours?

5- Six men can paint a house in four days. How long it would take to paint the house if three men are employed?

6- A manager plans to produce *100* bicycles with the help of *25* persons working *4* hours daily. How many bicycles can be made by *40* persons if they work *3* hours daily?

7- A factory makes *560* fans in *7* days with the help of *20* machines. How many fans can be made in *12* days with the help of *18* machines?

8- A factory makes *600* soaps in *9* days with the help of *20* machines. How many soaps can be made in *12* days with the help of *18* machines?

9- If the stay of *12* men for *28* days in a hotel costs *Rs.6720*. Find the cost for the stay of *8* men for *14* days in the hotel.

10- If the stay of *14* men for *8* days in a hotel costs *Rs. 22,400*. Find the cost for the stay of *7* men for *13* days.

11- *14* cows consume *63kg* of hay in *18* days. How many cows will eat *770 kg* of hay in *28* days at same rate?

12- Juice manufacturer produces *3000* bottles in a day employing *15* workers working *8* hours. Find the number of bottles manufactured when he employs *18* workers working *6* hours.

Review Exercise-1

1- Encircle the correct answer.

(i) 20 % of 600 is:

(a) 12 (b) 120 (c) 20 (d) 200

(ii) Fraction form of 70 % is:

(a) 7 (b) $\frac{7}{10}$ (c) $\frac{10}{7}$ (d) 7

(iii) $\frac{7}{20}$ in terms of percentage is:

(a) 35 % (b) 35 (c) 20 (d) 20 %

(iv) $\frac{1}{3}$ in terms of percentage is:

(a) 3 % (b) 1 % (c) 33 % (d) $33\frac{1}{3}\%$

(v) 0.13 as percentage is:

(a) 13 (b) 30 (c) 13 % (d) 10 %

(vi) In a ratio $a : b$, "a" is called:

(a) *extreme* (b) *antecedent*

(c) *mean* (d) *consequent*

(vii) In a ratio $a : b$, "b" is called:

(a) *extreme* (b) *antecedent*

(c) *mean* (d) *consequent*

(viii) In a proportion $a : b :: c : d$, a and d are called:

(a) *extremes* (b) *means*

(c) *antecedents* (d) *consequents*

(ix) In a proportion $a : b :: c : d$, b and c are called:

(a) *means* (b) *extremes*
(c) *consequents* (d) *antecedents*

(x) Lowest form of $75 : 95$ is:

(a) $15 : 17$ (b) $15 : 19$
(c) $19 : 15$ (d) $17 : 15$

2- Fill in the blanks.

(i) $30\ \%$ of 1500 is ____________

(ii) Fraction form of $15\ \%$ is ____________

(iii) $\frac{7}{25}$ in terms of percentage is ____________

(iv) $\frac{2}{3}$ in terms of percentage is ____________

(v) 0.29 as percentage is ____________

(vi) In a ratio $a : b$ "a" is called ____________

(vii) In a ratio $a : b$ "b" is called ____________

(viii) In a proportion $a : b :: c : d$, a and d are called ____________

(ix) In a proportion $a : b :: c : d$, the product of extremes is equal to the product of____________

(x) The simplest form of $\frac{2}{3} : \frac{3}{5}$ is____________

3- A railway train carries *800* passengers, *55%* passengers are men, *15%* are children. What is the percentage of women?

4- Azeem spends *25%* of his income on house rent, *60%* of the remaining on household expenditures. If he saves *Rs.2100*, what is his total income?

5- In a school there are *220* student chairs, *110* student tables, *50* staff chairs and *30* staff tables. Find the ratio of the following.

(i) Students chairs to students tables.

(ii) Students chairs to staff chairs.

(iii) Students tables to staff tables.

6- Two angles in a triangle are 48° and 60°. Find the ratio of the third angle to the sum of the first two angles.

7- *8* persons can do a job in *24* days, if *4* more persons joined them, how much time they will take to complete the same job?

8- The stay of *18* students for *36* days in a hostel costs *Rs.58320*. Find the cost for the stay of *9* students for *12* days.

SUMMARY

- Percentage means out of hundred.
- A comparison between the two like quantities is called a ratio.
- In a ratio $a:b$, "a" is called the antecedent.
- In a ratio $a:b$, "b" is called the consequent.
- The equality of two ratios is called proportion.
- If $a:b::c:d$, then "a" and "d" the first and fourth terms are called extremes and "b" and "c" the *2nd* and *3rd* terms are called means
- The relation between two ratios in which an increase in one quantity causes a proportional increase in the other quantity or decrease in one quantity causes a decrease in the other quantity is called direct proportion.
- The relationship between two ratios in which increase in one quantity causes a proportional decrease in the other quantity or vice versa is called inverse proportion.
- The relationship between two or more proportions is called a compound proportion.

UNIT 2

ZAKAT, USHR AND INHERITANCE

- **Zakat**
- **Ushr**
- **Inheritance**

After completion of this unit, the students will be able to:

- Know 'Nisab', both in 'tola' and gram, on which 'Zakat' is due.
- Know the rate of Zakat.
- Calculate amount of Zakat in respect of assets owned by a person.
- Know the rate of 'Ushr' levied on land-owner/land-holder in respect of produce of the land.
- Calculate amount of Ushr in respect of produce of land.
- Solve real life problems involving Zakat and Ushr.
- Know the ratio of shares among legal inheritors of a property.
- Calculate amount of share of each legal inheritor of a property.

2.1 ZAKAT

Zakat is one of the five basic pillars of Islam. Zakat is a "trans/ repayment" which Sahib-e-Nisab Muslims pay at given rate by themselves or through the Islamic state to the poors and the needy in or after the month of Rajab.

2.1.1 Nisab

A Muslim who owns and keeps in his / her possession at least *7.5* tola *(86.1562 gm)* gold or *52.5* tola *(603gm)* silver or cash money to the equivalent value for one year is considered as Sahib-e-Nisab Muslim. He is required to pay Zakat at the prescribed rates given in Quran and Hadith.

Zakat is paid from two types of wealth i.e exposed and unexposed wealth.

Exposed wealth includes agricultural goods, camels, sheep, goats, minerals, business inventories etc, whereas unexposed wealth includes gold, silver, cash money, liquid assets etc.

2.1.2 Rate of Zakat

Zakat is paid from the exposed or unexposed wealth by a Sahib-e-Nisab Muslim. The rate of Zakat is *2.5%* or $\frac{1}{40}$ of the total value of the goods or cash amount.

However capital goods like machinery equipment, raw material, factory building etc. are exempted from Zakat.

EXAMPLE-1

Calculate the amount of Zakat on an amount of Rs.5,00,000.

SOLUTION: Amount = Rs. 5,00,000

Rate of Zakat = 2.5 %

$$= 2.5 \times \frac{1}{100} = \frac{25}{10} \times \frac{1}{100}$$

$$= \frac{25}{1000}$$

$$\text{Amount of Zakat} = Rs.\frac{25}{1000}\times 5,00,000$$
$$= Rs.25\times 500$$
$$= Rs.12,500$$

EXAMPLE-2

Calculate the amount of Zakat on an amount of Rs.3,00,000, gold of weight 40 grams and silver of weight 500 grams, where as the rate of gold is Rs.3500 per gram and the rate of silver is Rs.400 per gram.

SOLUTION:

Cash amount = Rs. 3,00,000 .... (i)

Weight of gold = 40 grams

Rate of gold per gram = Rs. 3500

Amount of 40 grams of gold = Rs.3500 × 40

= Rs.1,40,000 .... (ii)

Weight of silver = 500 grams

Rate of silver per gram = Rs. 400

Amount of 500 grams of silver = Rs.400 × 500

= Rs.2,00,000 .... (iii)

From (i), (ii) and (iii)

Total amount on which Zakat is to be deducted is Rs.3,00,000 + Rs.1,40,000 + Rs.2,00,000 = Rs.6,40,000.

Rate of Zakat = 2.5 %

$$\text{Zakat on } Rs.6,40,000 = Rs.6,40,000\times\frac{2.5}{100}$$
$$= Rs.\frac{25\times 6,40,000}{1000}$$
$$= Rs.25\times 640$$
$$= Rs.16,000$$

2.2 USHR

Ushr is paid at the rate of $10\,\%$ from the agricultural products of the land which is irrigated by natural resources. However the rate of Ushr is $5\,\%$ on the agriculture products of the land which is irrigated by artificial sources, that is canals, tube-wells etc.

EXAMPLE-1

If the wheat crop is produced 40000kg by natural resources, calculate the amount of Ushr, if the price of wheat is Rs.950 per 40kg.

SOLUTION: Weight of wheat $= 40{,}000\text{kg}.$

Price of 40 kg wheat $= Rs.950$

Price of 1 kg wheat $= Rs.\frac{950}{40}$

Price of 40,000kg wheat $= Rs.40000\times\frac{950}{40}$

$= Rs.1000\times 950$

$= Rs.9{,}50{,}000$

Amount of Ushr $= Rs.\frac{10}{100}\times 9{,}50{,}000$

$= Rs.95{,}000$

EXAMPLE-2

Calculate the amount of Ushr on a rice crop of weight 3000kg produced by artificial sources, if the price of 40 kg rice is Rs.2000.

SOLUTION: Weight of rice crop $= 3000\text{kg}$

Price of 1 kg rice $= Rs.\frac{2000}{40}$

$= Rs.50$

Price of 3000 kg rice $= 3000\times 50$

$= Rs.1{,}50{,}000$

Amount of Ushr $= \frac{5}{100}\times 1{,}50{,}000$

$= Rs.7500$

2.3 INHERITANCE

When a person dies, then the assets left by that person are called inheritance, and it is distributed among his / her legal heirs. The principles of distribution of inheritance are given very clearly in Islam.

The following amounts are paid before distributing inheritance among heirs.

1- Payment of funeral expenses.

2- Payment of his / her debts.

3- Execution of his / her will.

2.3.1 Ratio of Shares Among Legal Inheritors of a Property

Firstly we determine which of the relatives of the deceased are entitled to inherit and secondly to determine the share of each heir.

In Islam there are total of twelve relations who inherit as legal heirs. We discuss ten of these in the following.

1- In case the husband dies, the share of the wife (widow) is one quarter $\left(\frac{1}{4}\right)$ in the absence of a child or agnatic* grandchild.

2- One eighth $\left(\frac{1}{8}\right)$ in the presence of a child or agnatic grandchild.

3- Two or more wives share equally in this prescribed share.

4- A son inherits a share equivalent to that of two daughters, i.e a brother inherits twice as much as a sister.

5- A grand son inherits twice as much as grand daughter and so on.

* Agnatic is from Agnate, which means, related on the father's side or through a male ancestor.

6- If the daughters are two or more than two, then for them two third of the inheritance.

7- If there is only a single daughter or agnatic grand daughter her share is a fixed one half.

8- Two or more daughters will totally exclude any grand daughter.

9- If there is one daughter and agnatic grand daughter, the daughter inherits one half $\left(\frac{1}{2}\right)$ share and the agnatic grand daughters inherit the remaining one sixth $\left(\frac{1}{6}\right)$, then making a total two third.

10- The husband inherits one half of the inheritance in case the wife dies.

> ***Remember That:***
>
> Distribution of remaining inheritance amongst the heirs according to Sharia is one of the four duties to be performed when a Muslim dies.

2.3.2 Calculate Amount of Share of Each Legal Inheritor of a Property

Let us calculate the amount of share of each legal inheritor with the help of following examples:

EXAMPLE-1

A person left a property of worth Rs. 24,00,000. Calculate the amount of share of his wife, a son and a daughter.

SOLUTION: *Amount of property* = *Rs. 24,00,000*

$$\text{Wife's share} = \frac{1}{8} \times 24,00,000$$

$$= Rs. 3,00,000$$

$$\text{Remaining amount} = Rs.\ 24,00,000 - 3,00,000$$
$$= Rs.21,00,000$$

$$\text{Ratio of son's share to the daughter} = 2:1$$
$$\text{Sum of the ratios} = 2+1$$
$$= 3$$

$$\text{Therefore daughter's share} = Rs.21,00,000 \times \frac{1}{3}$$
$$= Rs.\ 7,00,000$$
$$\text{Son's share} = 2 \times 7,00,000$$
$$= Rs.\ 14,00,000$$

EXAMPLE-2

A person left a property amounting Rs. 30,00,000. Calculate the amount of his wife, if he had not any child.

SOLUTION:

$$\text{Amount of property} = Rs.\ 30,00,000$$
$$\text{Wife's share} = 30,00,000 \times \frac{1}{4}$$
$$= Rs.\ 7,50,000$$

EXAMPLE-3

The amount of a property left by a deceased person is to be distributed among his daughters Rs. 3,30,000. If he had left three daughters only, what would be the share of each daughter?

SOLUTION:

$$\text{Amount left by deceased person} = Rs.\ 3,30,000$$
$$\text{Daughter's share} = 3,30,000 \times \frac{2}{3}$$
$$= Rs.110000 \times 2$$
$$= Rs.\ 2,20,000$$
$$\text{Share of each daughter} = 2,20,000 \times \frac{1}{3}$$
$$= Rs.\ 7,33,33.33$$

EXAMPLE-4

If a deceased person left a widow, a son and two daughters, calculate the share of each, if the wealth left by him amounts to Rs. 48,00,000.

SOLUTION: Amount of wealth left = Rs. 48,00,000

$$\text{Widow's share} = 48,00,000 \times \frac{1}{8}$$

$$= \text{Rs. } 6,00,000$$

$$\text{Remaining amount} = 48,00,000 - 6,00,000$$

$$= \text{Rs. } 42,00,000$$

$$\text{Ratio of son and daughters share} = 2 : 1 : 1$$

$$\text{Sum of the ratios} = 2 + 1 + 1$$

$$= 4$$

$$\text{Son's share} = \frac{2}{4} \times 42,00,000$$

$$= \text{Rs. } 2 \times 10,50,000$$

$$= \text{Rs. } 21,00,000$$

$$\text{Share of each daughter} = \frac{1}{4} \times 42,00,000$$

$$= \text{Rs. } 10,50,000$$

EXERCISE – 2.1

1- Calculate Zakat on gold amounting to *Rs.11,10,000.*

2- Calculate Zakat on silver amounting to *Rs.3,00,000.*

3- Calculate the amount of Zakat on *10* tola gold and *40* tola silver, if the rate of gold is *Rs.40,000* per tola and the rate of silver is *Rs.5000* per tola.

4- Calculate Zakat on gold of worth *Rs.8,00,000*, cash of amount *Rs.4,00,000* and silver of weight *50* tola *(Rs.5000 per tola).*

5- Calculate Ushr on a rice crop produced by natural resources amounting to *Rs.6,00,000.*

6- Calculate Ushr on a wheat crop amounting to *Rs. 3,50,000* produced by artificial resources.

7- Work out the share of each, if the inherited property amounting to *Rs. 7,50,000* is left by a deceased, who also left a widow, two son's and one daughter.

8- An amount of *Rs. 4,00,000* left as an inheritance is to be distributed among a widow and four daughters. Work out the share of each.

9- If a deceased left a property of worth *Rs. 15,00,000*, workout the property, if he left behind a widow

10- The inherited property amounting to *Rs. 20,00,000* is left by a deceased. He left behind a widow and two son's. Workout the share of each.

11- Asghar left a property of worth *Rs. 4,80,000*. He left behind a widow, three sons and four daughters. Calculate the share of each one.

12- Najeeb left a wealth amounting to *Rs. 4,00,000*. He left behind a widow, while they did not have any child. Find the share of Najeeb's widow.

Review Exercise-2

1- Encircle the correct answer.

(i) Zakat is deducted at a rate of:

(a) 2.5 % (b) 3.5 % (c) 4.5 % (d) 5.5 %

(ii) On a crop produced on natural resources, Ushr is deducted at a rate of:

(a) 2.5 % (b) 5 % (c) 10 % (d) 20 %

(iii) On a crop produced on artificial resources Ushr is deducted at the rate of

(a) *5 %* (b) *10 %* (c) *2.5 %* (d) *25 %*

(iv) Zakat on an amount of *Rs.100,000* is:

(a) *2500* (b) *25000* (c) *2000* (d) *15000*

*(v)*Ushr on a wheat crop produced on natural resources amounting Rs.1,50,000 is:

(a) *10,000* (b) *5000* (c) *15000* (d) *20000*

(vi) The share of a widow in inherited property is:

(a) $\frac{1}{4}$ (b) $\frac{1}{8}$ (c) $\frac{1}{3}$ (d) $\frac{1}{2}$

*(vii)*The share of a widow in the presence of a child or agnatic grand child is:

(a) $\frac{1}{4}$ (b) $\frac{1}{8}$ (c) $\frac{1}{2}$ (d) $\frac{1}{6}$

(viii) If there is only a single daughter or an agnatic grand daughter, her share is fixed:

(a) $\frac{1}{4}$ (b) $\frac{1}{6}$ (c) $\frac{1}{2}$ (d) $\frac{1}{8}$

(ix) If there are two or more than two daughters or one agnatic grand daughter then their share is:

(a) $\frac{2}{3}$ (b) $\frac{1}{4}$ (c) $\frac{1}{2}$ (d) $\frac{1}{8}$

(x) If there is one daughter and agnatic grand-daughters, their share are respectively:

(a) $\frac{1}{2},\frac{1}{6}$ (b) $\frac{1}{2},\frac{1}{3}$ (c) $\frac{1}{2},\frac{1}{4}$ (d) $\frac{1}{2},\frac{1}{8}$

2- Fill in the blanks.

(i) Zakat is deducted at a rate of______________.

(ii) On a crop produced on natural resources Ushr is deducted at a rate of______________.

(iii) On a crop produced on artificial resources Ushr is deducted at a rate of______________.

(iv) Zakat on an amount of *Rs.2,00,000* is______________.

(v) Ushr at a rate of *10 %* on amount of *Rs.1,00,000* is______________.

(vi) In an inherited property the share of a widow is______________.

(vii) In an inherited property the share of a widow in case of no child is______________

(viii) If there only a single daughter then her share in inherited property is______________.

(ix) The share of son and daughter in an inherited property is in the ratio______________.

(x) If there are two or more than two daughters, then their share in an inherited property is ______________.

3- Calculate Zakat on gold amounting *Rs.15,00,000*.

4- Calculate Ushr on a rice crop amounting *Rs.4,90,000* produced by artificial resources.

5- A deceased left a property of worth *Rs.45,00,000* . If he left behind a widow and two sons, work out the share of each.

6- Akram left a property of worth *Rs.48,00,000*. He left behind a window, three sons and four daughters. Calculate the share of each.

SUMMARY

- Zakat is one of the five basic pillars of Islam. Zakat is a "trans/ repayment" which Sabhib-e-Nisab Muslims pay at given rate by themselves or through the Islamic state to the poor and the needy once in a year.

- A Muslim who owns and keeps in his / her possession at least *7.5* tola *(86.1562 gm)* gold or *52.5* tola *(603gm)* silver or cash money to the equivalent value for one year is considered a Sahib-e-Nisab Muslim.

- Exposed wealth includes agriculture goods, camels, sheep, goats, minerals, business inventories etc.

- Unexposed wealth includes gold, silver, cash money, liquid assets etc.

- Rate of Zakat is *2.5 %* of the total value of the goods or money.

- Ushr is a tax paid at the rate of *10 %* from agriculture products of land which is irrigated by natural resources and *5 %* by artificial.

- When a person dies, then the assets left by that person are called inheritance.

UNIT

3

BUSINESS MATHEMATICS

- Profit and Loss
- Discount
- Business Partnership

After completion of this unit, the students will be able to:

- Know the:
  - Cost price *(CP)* as the price, an article is purchased for.
  - Selling price *(SP)* as the price, an article is sold for.
- Identify the following relations regarding profit (when $SP>CP$):
  - $\text{Profit} = SP - CP$,
  - $SP = \text{Profit} + CP$,
  - $CP = SP - \text{Profit}$,
  - $\text{Profit}\,\% = \frac{\text{Profit}}{CP} \times 100$,
  - $\text{Profit} = \frac{CP \times \text{Profit}\,\%}{100}$,
  - $SP = CP \times \left(\frac{100 + \text{Profit}\,\%}{100}\right)$,
  - $CP = \frac{100 \times SP}{100 + \text{Profit}\,\%}$,
- Identify the following relations regarding loss (when $SP<CP$):
  - $\text{Loss} = CP - SP$,
  - $SP = CP - \text{Loss}$,
  - $CP = \text{Loss} + SP$,
  - $\text{Loss}\,\% = \frac{\text{Loss}}{CP} \times 100$,
  - $\text{Loss} = \frac{CP \times \text{Loss}\,\%}{100}$,
  - $SP = CP \times \left(\frac{100 - \text{Loss}\,\%}{100}\right)$,
  - $CP = \frac{100 \times SP}{100 - \text{Loss}\,\%}$.
- Solve real life problems involving profit and loss.
- Recognize marked price (*MP*) or list price of an article.
- Identify the following relations regarding discount:
  - $\text{Discount} = MP - SP$,
  - $SP = MP - \text{Discount}$,
  - $\text{Discount}\,\% = \frac{\text{Discount}}{MP} \times 100$,
  - $SP = MP \times \left(\frac{100 - \text{Discount}\,\%}{100}\right)$,
  - $MP = \frac{100 \times SP}{100 - \text{Discount}\,\%}$.

3.1 PROFIT AND LOSS

Traders purchase and sell goods and services. The traders may earn profit or incur losses. We use arithmetic to calculate cost of goods purchased and profit or loss incurred by traders.

In order to calculate costs, profit and loss, in our daily life, we use arithmetics. This is being done by purchasing some articles from different shops every day.

3.1.1 Cost Price and Selling Price

The price at which a particular item is purchased by shopkeepers is called the cost price. It is denoted by "*CP*".

The price at which an article is sold out to the customer by the shopkeeper is called the selling price. Selling price is denoted by "*SP*".

3.1.2 Profit

If the selling price of an article is greater than its cost price, then the profit is earned. Profit is denoted by "*P*". The following mathematical relations exist between profit, selling price, cost price and profit percentage.

$$Profit = Selling\ Price - Cost\ Price$$

$$P = SP - CP$$

$$SP = P + CP$$

$$Profit\,\% = \frac{Profit}{CP} \times 100 \Rightarrow Profit = \frac{CP \times Profit\,\%}{100}$$

Here $SP = Profit + CP$

Therefore $$SP = \frac{CP \times Profit\,\%}{100} + CP$$

or $$SP = CP \times \left(\frac{Profit\,\% + 100}{100}\right)$$

and $$CP = SP \times \left(\frac{100}{100 + Profit\,\%}\right)$$

EXAMPLE-1

A bicycle was purchased for Rs.3450 and sold for Rs.3850. Find the profit percentage.

SOLUTION:

$$CP \text{ of the bicycle} = Rs.\,3450$$

$$SP \text{ of the bicycle} = Rs.\,3850$$

Therefore, $Profit = SP - CP$

$$= Rs.3850 - Rs.3450$$

$$= Rs.400$$

$$Profit\,\% = \left(\frac{Profit \times 100}{CP}\right)\%$$

$$= \left(\frac{400 \times 100}{3450}\right)\%$$

$$= 11.59\,\% \approx 11.6\,\%$$

EXAMPLE-2

A trader earns a profit of 20 % by selling a chair for Rs.2700. Find the cost price of the chair.

SOLUTION: Let the cost price of the chair be Rs.100.

Then profit $= Rs.\,20$ (i.e. 20%)

Hence $SP = CP + P$

Therefore $SP = 100 + 20 = Rs.\,120$

$$\text{If } SP \text{ is } Rs.120, \text{ then } CP = Rs.\,100$$

$$\text{If } SP \text{ is } Re.1, \text{ then } CP = Rs.\frac{100}{120}$$

$$= Rs.\frac{5}{6}$$

$$\text{If } SP \text{ is } Rs.2700, \text{ then } CP = Rs.\left(2700 \times \frac{5}{6}\right)$$

$$= Rs.\frac{13500}{6}$$

$$Cost\ price = Rs.2250$$

Alternate method:

We have the following formula to find cost price

$$CP = \left(\frac{100}{100 + Profit\,\%}\right) \times SP$$

$$= Rs.\left(\frac{100}{100+20}\right) \times 2700$$

$$= Rs.\frac{100 \times 2700}{120}$$

$$= Rs.\frac{13500}{6}$$

$$Cost\ price = Rs.2250$$

EXAMPLE-3

If a television is purchased for Rs.6590 and sold for Rs.6850. Find the profit percentage.

SOLUTION: Given

$$CP = Rs.6590$$

$$SP = Rs.6850$$

$$Profit = 6850 - 6590$$

$$Profit = Rs.260$$

$$Profit\,\% = \left(\frac{Profit \times 100}{CP}\right)\%$$

$$= \left(\frac{260 \times 100}{6590}\right)\%$$

$$= 3.94\% \approx 4\%$$

EXAMPLE-4

If the selling price of 10 articles is equal to the cost price of 11 articles. Find the profit percentage.

SOLUTION: Let the cost price of each article be Re.1.

$$\text{Then cost price of 10 articles} = Rs.10$$

$$\text{Then cost price of 11 articles} = Rs.11$$

$$SP \text{ of 10 articles} = CP \text{ of 11 articles}$$

$$\text{Therefore, selling price of 10 articles} = Rs.11$$

$$Profit = SP - CP$$

Therefore, $$Profit = Rs.(11-10) = Re.1$$

$$Profit\,\% = \left(\frac{Profit}{CP}\times 100\right)\%$$

Thus, $$Profit\,\% = \left(\frac{1}{10}\times 100\right)\%$$

$$= 10\%$$

EXAMPLE-5

By selling 100 oranges, a vendor gains the selling price of 20 oranges. Find the profit percentage.

SOLUTION: Let the cost price of each orange be Re.1.

Now $$SP \text{ of 100 oranges} = CP \text{ of 100 oranges} + profit$$

$$= CP \text{ of 100 oranges} + SP \text{ of 20 oranges}$$

$$\text{Therefore } SP \text{ of 80 oranges} = CP \text{ of 100 oranges.}$$

$$= Rs.100$$

$$CP \text{ of 80 oranges} = Rs.80$$

Therefore $$Profit = Rs.(100-80) = Rs.20$$

Thus $$Profit\,\% = \left(\frac{Profit}{CP}\times 100\right)\% = \left(\frac{20}{80}\times 100\right)\%$$

$$= \left(\frac{1}{4}\times 100\right)\%$$

$$= 25\%$$

EXAMPLE-6

A book is sold for Rs. 650 at a profit of 30%. Find the cost price.

SOLUTION:

If there is a profit of 30% and the cost price is CP.

Then $$CP = \frac{100 \times SP}{100 + profit\,\%}$$

$$= \frac{100 \times 650}{100 + 30}$$

$$= \frac{65000}{130}$$

$$= Rs. 500$$

Thus cost price of book is Rs. 500

EXAMPLE-7

On an electronic shop a shopkeeper sells a room heater for Rs.2100 gaining $\frac{1}{6}$ of its cost price. Find his profit percentage.

SOLUTION:

Let $$CP = Rs.\,x$$

$$Profit = Rs.\frac{x}{6}$$

Therefore, $$SP = Profit + CP = Rs.\left(\frac{x}{6} + x\right)$$

$$= Rs.\left(\frac{7x}{6}\right)$$

Therefore, $$\frac{7x}{6} = 2100$$

$$x = \frac{2100}{7} \times 6$$

$$= 300 \times 6$$

$$= Rs.1800$$

Therefore, $CP = Rs.1800,\ SP = Rs.2100$

$$Profit = Rs.(2100 - 1800) = Rs.300$$

$$Profit\,\% = \left(\frac{P}{CP} \times 100\right)\%$$

$$Profit\,\% = \left(\frac{300}{1800} \times 100\right)\%$$

$$= \left(\frac{1}{6} \times 100\right)\%$$

$$= 16.66\,\%$$

Thus, $Profit\,\% = 16.67\,\%$

EXAMPLE-8

A shopkeeper bought 100 hockey balls for Rs.40 each. He sells 20 of them at a profit of 5%. At what profit percent must he sell the remaining so as to get profit 20% on the whole?

SOLUTION: $CP \text{ of } 20 \text{ hockey balls} = Rs.(40 \times 20)$

$$= Rs.800$$

$$Profit \text{ on } 20 \text{ balls} = 5\,\%$$

$$SP \text{ of } 20 \text{ balls} = Rs.\left(\frac{105}{100} \times 800\right)$$

$$= Rs.840$$

Now, $CP \text{ of } 100 \text{ balls} = Rs.(40 \times 100)$

$$= Rs.4000$$

$$Required\ profit = 20\,\%$$

$$Required\ SP = Rs.\left(\frac{120}{100} \times 4000\right)$$

$$= Rs.4800$$

$$\text{Therefore, desired SP of 80 balls} = Rs.(4800 - 840)$$
$$= Rs.3960$$
$$\text{CP of 80 balls} = Rs.(40 \times 80)$$
$$= Rs.3200$$
$$\text{required gain on 80 balls} = Rs.(3960 - 3200)$$
$$= Rs.760$$
$$\text{(Required profit)\%} = \left(\frac{Profit}{CP} \times 100\right)\% = \left(\frac{760}{3200} \times 100\right)\%$$
$$= \left(\frac{760}{32}\right)\%$$
$$= \left(\frac{190}{8}\right)\%$$
$$= 23.75\%$$

3.1.3 Loss

If the sales price of an article is less than its cost price, then there is always a loss. The following mathematical relations exist between loss, selling price, cost price and loss percentage.

$$Loss = Cost\ Price - Selling\ Price$$
$$Loss = CP - SP$$
$$SP = CP - Loss$$
$$CP = Loss + SP$$
$$Loss\% = \frac{Loss}{CP} \times 100.$$
$$Loss = \frac{CP}{100} \times Loss\%$$
$$SP = CP \times \left(\frac{100 - Loss\%}{100}\right)$$
$$CP = \frac{100 \times SP}{100 - Loss\%}$$

EXAMPLE-1

Daniyal buys 6 sweets at a rupee and sells them, 8 sweets for a rupee. Find his loss percentage.

SOLUTION: *L.C.M of 6 and 8 is 24.*

Let us suppose that Daniyal buys 24 sweets.

$$CP\ of\ 24\ sweets = Rs.\left(\frac{1}{6}\times 24\right) = Rs.4$$

$$SP\ of\ 24\ sweets = Rs.\left(\frac{1}{8}\times 24\right) = Rs.3$$

$$Loss = CP - SP$$

$$= Rs.(4-3) = Re.1$$

Thus,
$$Loss\% = \frac{loss}{CP}\times 100$$

$$= \frac{1}{4}\times 100 = 25\%$$

EXAMPLE-2

A shopkeeper sold two radios at Rs. 1020 each, On one he gains 20% and on another he loses 20%.

How much does he gain or lose in the whole transaction?

SOLUTION: *In case of first radio:*

$SP = Rs.1020, \quad Profit = 20\%$

$$\text{Therefore, } CP = Rs.\left(\frac{100}{120}\times 1020\right)$$

$$= Rs.(10\times 85)$$

$$= Rs.850$$

In case of second radio:

$SP = Rs.1020 \quad , \quad Loss = 20\%$

$$\text{Therefore, } CP = Rs.\left(\frac{100}{80}\times 1020\right)$$

$$= Rs.\left(\frac{5}{4}\times 1020\right)$$

$$= Rs.(5\times 255)$$

$$= Rs.1275$$

Total cost of both the radios $= Rs.850 + Rs.1275$

$= Rs.2125$

Total sale price of both the radios $= Rs.(1020 \times 2)$

$= Rs.2040$

Loss in whole transction $= Rs.(2125 - 2040)$

$= Rs.85$

$$\text{Thus, } Loss\,\% = \left(\frac{85}{2125} \times 100\right)\%$$

$$= 4\%$$

$$Loss\,\% = 4\%$$

EXAMPLE-3

A bicycle dealer sells a bicycle at a profit of 8%. Had he sold it for Rs. 75 less he would have lost 2%. Find the cost price of the bicycle.

SOLUTION: Let the $CP = Rs.\ x$.

$$\text{Hence } SP = CP\left(\frac{Profit\,\% + 100}{100}\right)$$

when $SP > CP$

$$\text{Therefore selling price at a profit of } 8\% = Rs.\left(\frac{108}{100} \times x\right)$$

$$= Rs.\left(\frac{27}{25}x\right)$$

$$SP = CP\left(\frac{100 - \%\,Loss}{100}\right), \text{ when } SP < CP$$

$$SP \text{ at a loss of } 2\% = Rs.\left(\frac{98}{100} \times x\right) = Rs.\frac{49}{50}x$$

$$\text{Difference between the selling prices} = Rs.\left(\frac{27}{25}x - \frac{49}{50}x\right)$$

$$= Rs.\left(\frac{54x - 49x}{50}\right)$$

$$= Rs.\frac{5x}{50} = Rs.\frac{x}{10}$$

But $Rs.\frac{x}{10} = 75$ (given)

$\Rightarrow x = 75 \times 10$

$\Rightarrow x = Rs.750$

Thus, the cost price of the bicycle is Rs. 750.

EXAMPLE-4

A boy bought a book for Rs. 575 and sold it for Rs. 320. What was his loss percentage?

SOLUTION:

$$\text{CP of the book} = Rs.\,575$$
$$\text{SP of the book} = Rs.\,320$$
$$\text{Loss} = CP - SP$$
$$= Rs.\,(575 - 320)$$
$$= Rs.\,255$$
$$\text{Loss percentage} = \left(\frac{loss}{cost\ price} \times 100\right)\%$$
$$= \left(\frac{255}{575} \times 100\right)\%$$
$$= \left(\frac{51}{115} \times 100\right)\%$$
$$= \left(\frac{5100}{115}\right)\% \quad = 44.34\%$$

3.1.4 Real Life Problems

In our daily life when we go to the market to purchase different sort of items like books, cloth, grocery, ready-made garments, electronics etc., we experience about CP, SP, Profit and Loss.

Let us consider the following examples for this purpose.

EXAMPLE-1

A shopkeeper sells a fan for Rs. 1520. At what price should he sell it to get a profit of 15%?

SOLUTION: *Let the cost price is Rs. 100. Then a profit of 15% means, that the selling price is:*

$$Rs.\,(100 + 15) = Rs.\,115$$

$$\text{when } CP = Rs.\,100, \quad \text{then} \quad SP = Rs.\,115$$
$$\text{when } CP = Re.\,1, \quad \text{then} \quad SP = Rs.\,\frac{115}{100}$$
$$\text{when } CP = Rs.\,1520, \quad \text{then} \quad SP = Rs.\,\frac{115}{100} \times 1520$$
$$= Rs.\,23 \times 76 = Rs.\,1748$$

EXAMPLE-2

While selling a shirt for Rs. 960, the shopkeeper lost 20%. For what price should he sell to get 35% profit?

SOLUTION: Let $CP = Rs.\ 100$

$$Loss = Rs. 20$$

$$SP = Rs.(100-20)$$

$$= Rs. 80$$

when, SP is Rs. 80 then $CP = Rs. 100$

when, SP is Re. 1 then $CP = Rs.\frac{100}{80}$

when, SP is Rs. 960, then $CP = Rs.\frac{100}{80}\times 960$

$$= Rs. 100 \times 12$$

$$= Rs. 1200$$

Thus the CP of the shirt is Rs. 1200.

Again let the $CP = Rs.\ 100$

Profit 35% means, $SP = Rs.\ 135$

when, $CP = Re. 1$, then $SP = Rs.\frac{135}{100}$

when, $CP = Rs. 1200$, then $SP = Rs.\frac{135}{100}\times 1200$

$$= Rs. 1620$$

Hence the shirt should be sold for Rs. 1620 to make a profit of 35%.

EXERCISE – 3.1

1- Find the SP, when

(i) $CP = Rs.\ 950,\ Profit = 10\ \%$ (ii) $CP = Rs.\ 1540,\ Loss = 5\%$

(iii) $CP = Rs.\ 9600,\ Profit = 10\ \%$ (iv) $CP = Rs.\ 126000,\ Loss = 5\%$

(v) $CP = Rs.\ 480,\ Profit = 3\ \%$ (vi) $CP = Rs.\ 760,\ Loss = 4\%$

2- Haris purchased a car for $Rs.248000$ and spent $Rs.12000$ on its denting and painting. He sold that at a profit of $5\ \%$. What did the customer pay to Haris?

3- Find the CP, when

(i) $SP = Rs.\ 672,\ Profit = 5\ \%$ (ii) $SP = Rs.\ 851,\ Loss = 8\ \%$

(iii) $SP = Rs.\ 1755,\ Profit = 12\frac{1}{2}\%$ (iv) $SP = Rs.\ 2640,\ Loss = 12\ \%$

(v) $SP = Rs.\ 100,\ Profit = 33\frac{1}{2}\%$

4- A shop-keeper gains a profit of 7% by selling a dinner set for $Rs.3852$. If he sells it for $Rs.4050$, find his profit percentage.

5- The selling price of 12 articles is equal to the cost price of 15 articles. Find profit percentage.

6- Find the cost price, if a fan is sold for $Rs.1470$, to get a profit $\frac{1}{6}$ th of its cost price.

7- A man sold an almirah at a profit of $7\frac{1}{2}\%$, had he sold it for $Rs.209$, he would have lost 2%. For how much the man purchased it?

8- Three chairs are purchased at $Rs.450$ each. One of these is sold at a loss of $10\ \%$. At what price should the other two be sold so as to gain $20\ \%$ on the whole transaction?

3.2 DISCOUNT

A deduction offered on the marked price or the list price of goods by the seller to the purchaser is called discount.

3.2.1 Marked Price (MP) and List Price

The printed price on the tag or wrapper of the article is called marked price *(MP)*. The price of an article given in the list provided by the manufacturer to the trader is called list price *(LP)*.

$$Discount = List\ Price\ or\ Marked\ Price \times Discount\ \%$$

$$Discount = Marked\ Price - Sale\ Price.$$

$$= MP - SP$$

or $$Discount = List\ Price - Sale\ Price.$$

$$= LP - SP$$

3.2.2 Relations Regarding Discount

$$Discount = Marked\ Price - Sale\ Price$$

$$= MP - SP$$

$$Sale\ Price = Marked\ Price - Discount$$

$$SP = MP - Discount$$

$$Discount\ \% = \frac{Discount}{MP} \times 100$$

$$Sale\ Price = Marked\ Price \times \left(\frac{100 - Discount\ \%}{100}\right)$$

$$SP = MP \times \left(\frac{100 - Discount\ \%}{100}\right)$$

$$MP = \frac{100 \times SP}{100 - Discount\ \%}$$

3.2.3 Real Life Problems

EXAMPLE-1

The marked price of a toy is Rs.750 and 2% discount is offered on cash payment. What cash payment one should pay for the toy?

SOLUTION:

Marked price of the toy = Rs.750, Discount rate = 2%

$$\text{Discount} = (\text{Discount}\,\% \times MP) = Rs.\left(\frac{2}{100}\times 750\right)$$

$$= Rs.15$$

$$SP = MP - Discount$$

$$SP = Rs.(750-15) = Rs.735$$

Thus Rs. 735 should be paid to purchase the toy.

EXAMPLE-2

An article is sold for Rs.1000 after allowing a discount of 7% on the marked price. Find its marked price.

SOLUTION: Let the marked price be Rs.100

Discount allowed on it = 7% of Rs.100

$$= \frac{7}{100}\times 100$$

$$= Rs.7$$

$$\text{Selling Price} = \text{Marked Price} - \text{Discount}$$

$$\text{Selling Price} = Rs.(100-7) = Rs.93$$

If SP is Rs.93, its marked price = Rs.100.

If SP is Re.1, its marked price $= Rs.\dfrac{100}{93}$

If SP is Rs.1000, its marked price $= Rs.\left(\dfrac{100}{93}\times 1000\right)$

$$= Rs.1075.27$$

Thus, $MP = Rs.1075.27$

EXAMPLE-3

A television dealer marks a television with a price which is 20%more than the cost price and offers of 10 % discount on it. Find the profit percentage.

SOLUTION: Let the CP be Rs.100

$$MP = Rs.(100 + 20)$$
$$= Rs.120$$
$$10\% \text{ of } Rs.120 = \frac{10}{100} \times 120$$
$$\text{Discount} = Rs.12$$

Therefore, $SP = Rs.(120 - 12)$

$$= Rs.108$$

Thus, $\text{Profit Percentage} = (SP - CP)\% = (108 - 100)\%$

$$= 8\%$$

EXAMPLE-4

A shopkeeper offers a discount of 15% on the marked price. How much more the cost price must he mark on his goods to gain a profit of 19 %?

SOLUTION: Let the cost price be. Rs.100

Then $gain = 19\% \text{ of } Rs.100 = Rs.19$

Therefore $SP = Rs.(100 + 19)$

$$= Rs.119$$

When MP is Rs.100, then $SP = 100 - 15 = Rs.85$

If SP is Rs.85, then $MP = Rs.100$

If Re.1 is the SP, then $MP = Rs.\left(\frac{100}{85}\right)$

If Rs.119 is the SP, then $MP = Rs.\left(\frac{100}{85} \times 119\right)$

$$= Rs.140$$

Thus shopkeeper must mark his goods 40 % above the cost price.

EXAMPLE-5

During January sales, a departmental store offers a discount of 10% on marked prices. What is the purchase price of a dinner set with marked price as Rs. 8450?

SOLUTION: Discount of 10% on a MP of Rs. 8450

$$= MP \times 10\%$$
$$= 8450 \times \frac{10}{100}$$
$$= Rs.845$$

$$Purchase\ Price = Marked\ Price - Discount$$
$$= Rs.(8450 - 845)$$
$$= Rs.7605$$

EXAMPLE-6

A bicycle dealer offers a discount of 10% and still makes a profit of 26%. What is the actual cost of a bicycle, with marked price as Rs. 840 ?

SOLUTION: $MP = Rs.\ 840$

$$Discount = 10\% \times MP = Rs.\left(\frac{10}{100} \times 840\right) = Rs.84$$

$$SP = Rs.(840 - 84) = Rs.756$$

$$Profit = 26\%$$

$$CP = \frac{100}{100 + Profit\,\%} \times SP$$

Therefore, $CP = Rs.\left(\frac{100}{100+26} \times 756\right)$

$$= Rs.\left(\frac{100}{126} \times 756\right)$$
$$= Rs.(100 \times 6)$$
$$= Rs.600$$

EXERCISE – 3.2

1- Find the selling price, when,

(i) $MP = Rs.\ 728, Discount = 6\ \%$

(ii) $MP = Rs.\ 2760, Discount = 5\ \%$

(iii) $MP = Rs.\ 395.75, Discount = 8\ \%$

2- Find the marked price when,

(I) $SP = Rs.515.20, Discount = 8\ \%$

(ii) $SP = Rs.858, Discount = 12\ \%$

(iii) $SP = Rs.\ 2400, Discount = 4\ \%$

3- The marked price of a ceiling fan is *Rs.720*. It is sold for *Rs.684*. What percentage discount is being allowed?

4- The marked price of washing machine is *Rs.3640*. During sale season it is sold for *Rs.3367*. Find the discount percentage.

5- The marked price of a book is *Rs.480*. The shopkeeper offers a discount of *10 %* and still gains *8%*. Find the price at which the shopkeeper purchased it.

6- A trader marks his goods in such a way that after allowing a discount of *10 %*, he gains *15%*. If an article costs him *Rs.720*, what is its marked price?

7- The list price of a *TV* is *Rs.12600*. A discount of *5 %* is allowed on it. Further for cash payment a second discount of *2 %* is given. How much cash payment is to be made for buying it?

8- If *15 %* discount on *MP* of a heater is allowed and still makes a profit of *2 %*. If it is sold on *MP*, what is profit percentage?

3.3.1 BUSINESS PARTNERSHIP

An association of two or more persons to carry on a business for the purpose of making profit is called partnership. Partnership can be classified in two types, Simple Partnership and Compound Partnership.

When capital of partners is invested for the same length of time, the partnership is called simple. In such cases, profit or loss is distributed in proportion to the amount of capital invested by each partner.

When the capital of partners whether equal or unequal are invested for different lengths of time, the partnership is called compound. In such cases, profit or loss is distributed in accordence with the products of the capital and the periods of their investments.

3.3.2 Profit among the Partners

In the following examples we show, how the profit among the partners is distributed.

EXAMPLE-1

Aslam, Anwar and Akram earned a profit of Rs.2,50,000 from a business. If their investments in the business are of ratio 4:7:14 respectively. Find the profit of each.

SOLUTION: Given ratios = 4:7:14

Sum of the ratios = 4+7+14

= 25

Profit earned = Rs. 2,50,000

Profit earned by Aslam $= \frac{4}{25} \times 2,50,000 = \text{Rs. } 40,000$

Profit earned by Anwar $= \frac{7}{25} \times 2,50,000 = \text{Rs. } 70,000$

Profit earned by Akram $= \frac{14}{25} \times 2,50,000 = \text{Rs. } 1,40,000$

EXAMPLE-2

The shares of three partners in a business are in the ratio 2:3:5. If they suffered a loss of Rs. 10,00,000 in the business, what was the share of each individual in the loss incurred?

SOLUTION: Given ratios = 2:3:5

Sum of the ratios = 2+3+5

= 10

Total loss = Rs. 10,00,000

Loss of the first share holder $= \frac{2}{10} \times 10,00,000 = Rs.\,200,000$

Loss of the second share holder $= \frac{3}{10} \times 10,00,000 = Rs.\,300,000$

Loss of the third share holder $= \frac{5}{10} \times 10,00,000 = Rs.\,500,000$

EXAMPLE-3

Rs. 3720 are to be divided into three shares in such a way that 1st share would be double, triple to the 2nd and 5 times to the 3rd are equal.

SOLUTION: Let the 1st, 2nd and 3rd shares be x, y and z respectively. According to the condition of the question.

$$2x = 3y = 5z$$

$$\frac{2x}{30} = \frac{3y}{30} = \frac{5z}{30} \quad \text{(dividing by 30, the L.C.M of 2.3 and 5)}$$

$$\frac{x}{15} = \frac{y}{10} = \frac{z}{6}$$

Therefore $x : y : z = 15 : 10 : 6$

Sum of ratios $= 15 + 10 + 6$
$= 31$

1st share $= Rs.3720 \times \frac{15}{31}$
$= Rs.120 \times 15$
$= Rs.1800$

2nd share $= Rs.3720 \times \frac{10}{31}$
$= Rs.120 \times 10$
$= Rs.1200$

3rd share $= Rs.3720 \times \frac{6}{31}$

$= Rs.120 \times 6$

$= Rs.720$

Check: $Rs.1800 + Rs.1200 + Rs.720 = Rs.3720$

EXAMPLE-4

Jamila and Alia enter into partnership and their shares are in the ratio of $\frac{1}{2}:\frac{1}{3}$. After 4 months, Jamila withdraws half of her capital and after 8 months more, a profit of Rs. 500 is divided. What is Jamila's share of profit?

SOLUTION: Ratio of Profit $= \frac{1}{2}:\frac{1}{3} = 3:2$

They must put their capital in the same ratio. If Jamila puts Rs.300, then Alia puts Rs.200. After 4 months Jamila withdraws half of her capital.

After 8 months profit earned $= Rs.500$

Jamila's investment for 4 months $= 300 \times 4$

$= Rs.1200$

Jamila's investment for the next 8 months $= \frac{1}{2} \times (300) \times 8$

$= Rs.150 \times 8$

$= Rs.1200$

Jamila's investment for 12 month $+ Rs.1200 = Rs.1200$

$= Rs.2400$

Alia's investment for 12 month $= 200 \times 12$

$= Rs.2400$

Since, the investment for each is the same,

thus Jamila's profit in $Rs.500 = \frac{1}{2} \times 500$

$= Rs.250$

EXAMPLE-5

Umer and Ali purchased a plot with an investment of Rs. 3,00,000 and Rs.5,00,000 respectively. On selling the plot they got a profit of Rs. 2,20,000. Find the share of each in profit.

SOLUTION:

	Umer invested		Ali invested
	3,00,000	:	5,00,000
	3	:	5

Ratio of profit $= 3 : 5$

Sum of the ratios $= 3 + 5 = 8$

Total profit $=$ Rs. 2,20,000

$$\text{Umer's share in profit} = \frac{2,20,000}{8} \times 3$$

$$= 27500 \times 3$$

$$= \text{Rs. } 82,500$$

$$\text{Ali's share in profit} = \frac{2,20,000}{8} \times 5$$

$$= 27500 \times 5$$

$$= \text{Rs. } 1,37,500$$

EXERCISE – 3.3

1- Distribute *Rs. 200,000* as a profit in a business regarding three persons, if their shares are in the ratio *3:2:5*.

2- If Ali, Daniyal and Abdullah earned *15 %* profit against an investment of *Rs. 750,000*. Find the profit of each if their shares are in the ratio *2:3:5* .

3- Distribute *Rs. 720* as profit amongst three people, so that their shares are in the ratio *3:4:5* .

4- Three persons invested an amount of *Rs. 3,000,000* in a business with shares ratio *2:3:7*. They earned a profit of *Rs. 600,000*. If they are interested to wind up their business, what amount every share holder would get?

5- Three members of a firm distribute the profit *Rs. 67,200* among themselves in the ratio *2:3:7*. What is the biggest share of the profit?

6- A sum of money is divided among three persons, *A,B* and *C* in the ratio *10:7:5 respectively*. If "*B*" gets *Rs. 14* more than "*C*". How much will "*A*" get and what is the total sum of money?

Review Exercise-3

1- **Encircle the correct answer.**

i. Profit is earned when:

(a) $SP = CP$ (b) $SP < CP$ (c) $SP > CP$ (d) *none of these*

ii. Loss is there when:

(a) $SP = CP$ (b) $SP < CP$ (c) $SP = MP$ (d) $SP > CP$

iii. Profit % = ? where $SP > CP$

(a) $\frac{profit}{CP}$ (b) $\frac{profit}{CP} \times 100$

(c) $\frac{CP \times profit\,\%}{100}$ (d) $\frac{100 \times SP}{100 + profit\,\%}$

iv. $SP = ?$ where $SP > CP$

(a) $Profit - CP$ (b) $\left(\frac{100 + profit\,\%}{100}\right) \times CP$

(c) $CP - Loss$ (d) $\frac{CP \times loss\,\%}{100}$

v. $CP = ?$

(a) $\frac{100 \times SP}{100 + profit\,\%}$ (b) $loss - SP$

(c) $MP + discount$ (d) $\frac{discount \times 100}{MP}$

2- Fill in the blanks.

i. The price at which a particular item is purchased is called________

ii. The price at which an article is sold out is called ________

iii. When $SP > CP$, $CP = SP - ?$________

iv. When $SP < CP$, $Loss\ \% =$________

v. $MP = \dfrac{100 \times SP}{?}$ ________

3- A shopkeeper gains a profit of 8% by selling a washing machine for $Rs.12000$. If he sells it for $Rs.10,500$, find his loss percentage.

4- If there is a 10% discount on marked price of a television and still makes a profit of 5%. If it is sold in marked price, what is profit percentage?

5- Distribute $Rs.33,000$ as a profit in a business regarding three persons, if their shares are in the ratio $3:5:3$.

6- Three members of a firm distribute the profit amounting $Rs.1,44,000$ among themselves in the $3:4:5$.

(i) What is the biggest share of the profit?

(ii) What is the smallest share of the profit?

SUMMARY

- The price at which a particular item is purchased is called cost price. It is denoted by "*CP*".
- The price at which an article is sold out is called the sale price. It is denoted by "*SP*".
- If the selling price of an article is greater than its cost price, then the difference of these two is the profit earned. It is denoted by "*P*".
- If the selling price of an article is less than its cost price, then the difference of these two is the loss. It is denoted by "*L*".
- Some times a rebate is declared on the selling price of an article, this rebate is called the discount.
- The price tagged on a card of each and every article in a shop is known as the marked price, It is denoted by "*MP*".

UNIT

4

FINANCIAL MATHEMATICS

- Commercial Banking
- Exchange of Currencies
- Profit / Markup
- Insurance
- Leasing / Financing

After completion of this unit, the students will be able to:

- Know commercial bank deposit and types of a bank account (PLS saving bank account, current deposit account, PLS term deposit account and foreign currency account.)
- Describe negotiable instruments like cheque, demand draft and pay order.
- Explain on-line banking, transactions through ATM (Auto Teller Machine), debit card and credit card (Visa and Master).
- Convert the value of a given amount of the currency of one country in terms of another currency.
- Calculate:
  - The profit/markup,
  - The principal amount,
  - The profit/markup rate,
  - The period.
- Solve problems related to commercial banking and national saving schemes.
- Define insurance in its simple terms.
- Know life insurance and vehicle insurance.
- Solve simple real life problems regarding purchase of life and motor vehicle insurance.
- Know
  - Leasing/financing of motor vehicle,
  - Down payment,
  - Motor vehicle insurance,
  - Processing charges,
  - Repayment in monthly installments.
- Solve problems related to leasing/financing of motor vehicle under different conditions.

4.1 COMMERCIAL BANKING

A banking business which is related to the accepting of deposits, advancing loans and undertakes other services for its clients is called commercial banking. Bank collects the idle savings of people and firms in the form of deposits in different accounts.

4.1.1 Bank Deposit and Types of Accounts

There are three major types of accounts which can be maintained with banks to keep the deposit or surplus funds. These are explained below:

1- **Current Account:** Very popular accounts with high degree of liquidity.

2- **Saving Account:** An important source of funds for the banks.

3- **Fixed Account:** An attractive source of fund for long term lending and investment purposes.

Current Account

A Current Account or demand deposit is running account which continuously remains in operation due to its liquidity. It is used by a customer to transfer and withdraw funds on demand without prior notice to the bank. The bank is bound to honour the cheques subject to availability of sufficient funds.

In Pakistan, the current account can be opened with a minimum amount ranging from *Rs. 1,000 to 10,000* with or without minimum balance maintenance requirements as specified by the bank. Since the funds placed in this account are for very short period, so interest or profit is not paid by bank on this account.

Saving Account

Saving account, as the name suggests, is meant to encourage thrift and promote saving among the persons of small means. The bank pays nominal interest half yearly on the basis of monthly balance.

The depositors are normally allowed to withdraw a limited amount of money without any prior notice but for withdrawal of large amount from such an account, a prior notice of 7 to 15 days in writing is required to be served to the bank. In order to mobilize savings and accommodate the clients, the bank normally waives off the notice period.

In view of the withdrawal frequency, bank keeps a minimum amount in reserve to meet the customer's demand and the remaining funds of saving accounts can safely be invested by the bank in any profit oriented venture/ schemes in the form of loans and advances.

PLS Saving Account

In Pakistan, the profit and loss sharing (PLS) saving account was introduced in January, 1982. PLS Saving account can be opened with small amount (normally not less than Rs. 100). A credit balance of Rs.100 is eligible for sharing profit and loss of the bank. Withdrawal of small amount from PLS account is allowed but for the whole amount a prior notice is required, if so desired by the bank. The profit earned & loss sustained on PLS saving account will be credited/debited as determined by the bank on the basis of its net working result at the end of each half year/full year, depending upon the mode of payment of profit.

Fixed / Time Deposit Account

Fixed or time deposit account, as the name implies, are deposits kept with a bank in an account for a certain period of time ranging from 3 months to 5 years. Time deposit is kept in the bank by the customers to earn profit. On maturity of the time deposit, the bank pays the principal amount along with profit of the stipulated period to the holders. This deposit is not payable on demand like the current deposits but can only be withdrawn by the depositors on expiry of the specific period for which the funds have been fixed. The rate of profit on fixed deposit is comparatively higher than Saving Deposit. The longer the duration of deposit, the higher is the rate of profit and vice versa.

The bank on receipt of funds for time deposit issues a receipt or a specially printed form as an evidence to the deposit holder. Normally such deposit holder is not issued cheque book for funds withdrawal. After the expiry of the fixed period, the depositor presents the receipt duly discharged and gets the amount in cash or gets the same transferred to his account. If the depositor needs funds before its maturity date, the bank usually obliges the customer with or without making partial amount of payable profit proportionately.

The depositor may also avail financing facility on nominal markup rate, normally *1* to *2%* over the deposit rate, from the bank against the security of time deposit. In case of demand deposits, the bank has to keep higher reserve ratio to meet the depositor's liabilities. But for the time deposit the reserve ratio is quite small.

Foreign Currency Accounts

A foreign currency account is the account maintained with the bank in foreign currency, like Dollars, Pounds, Euro etc. These accounts are maintained, operated in line with the instructions of State Bank of Pakistan. All the banks or authorized dealers may without prior approval of the State Bank of Pakistan, open foreign currency account of Pakistani national residents in or outside Pakistan, including those having a dual nationality. All foreign nationals, firms and companies established / incorporated and functioning in Pakistan including those having foreign share holding can open foreign currency accounts. However, airlines and shipping companies operating in or through Pakistan cannot open foreign currency account.

Foreign currency accounts can be operated in Current, Saving, Fixed accounts. Rate of profit in a foreign currency account is very nominal as compared to local currency accounts. Foreign currency accounts can be operated by remittances received from abroad and travelers cheques issued outside Pakistan. Accounts can be maintained and payment can be made in any currency of choice of the account holder. Credit card facility can be obtained by the account holders up to the extent that they can utilize their balance in or out side Pakistan.

The bank can mark lien on the foreign currency accounts in respect of banking facilities like credit card, bank guarantees and loan/credit etc. availed by the account holders in and outside Pakistan. Foreign currency accounts are exempted from zakat and taxes.

4.1.2 Negotiable Instruments

The word **"negotiable"** means transfer for consideration, whereas the word **"instrument"** means written documents creating right. So negotiable instruments means documents in writing which create a right in favour of some person and which is freely transferable by delivery or endorsement.

A negotiable instruments means a promissory note, a bill of exchange or cheque payable either to the order or bearer of the instrument.

Bill of Exchange

A Bill of Exchange is an instrument in writing containing an unconditional order, signed by the maker, directing a certain person to pay a certain amount only to or to the order of, a certain person or to the bearer of the instrument.

Parties of Bill of Exchange

Drawer: the person who draws the bill.

Drawee: the person in whose favour the bill is drawn.

Payee: the person to whom the payment is made.

Cheque

A bill of exchange drawn on a specified banker and not expressed to be payable, otherwise then on demand.

Parties of Bill of Cheque Drawer, Drawee & Payee

Pay Order

Pay order is like a cheque, issued by bank on the request of its customer or in payment of its own expenses or dues, drawn on itself, to pay a specified sum of money to the order of specified person. Pay orders are usually issued by the banks on receipt of full amounts involved, which means that it would not be returned unpaid due to lack of funds. It is also called banker's cheque or cashier's cheque.

Bank Draft

An order to pay money, drawn by one branch of a bank upon another branch of the same bank for an amount of money payable to or order of payee or on demand. The bank draft has an important advantage over a cheque as it would not be dishonored for lack of funds since it is funded instrument. The draft normally drawn from the branches located out of city. The bank charges nominal fee for preparation of bank draft where as pay orders are issued without any charges by most of the banks.

4.1.3 Online Banking

Electronic based banking service provided by devices such as ATM (Automated Teller Machine), POSs (Point of Sale), Automate Clearinghouse, Network, Internet or Wire. In other words, online bankin is a system for transmitting and executing instructions for bankin transactions in real time through electronic telecommunications an computerized links between bank and customer via telephone line o satellites or automated teller machines.

Under online banking system, the customer maintains account with an specific bank and can transfer and withdraw the amount from any branch of the same bank. If the customer has been issued a debit or credit card only then the customer can withdraw or transfer the funds through ATM linked with M.Net or One Link banks. Under online banking system instructions from the sender to remitting bank and from the remitting bank to the paying bank, are transmitted through electronic means of communications, funds received and payment takes place in real time instantly.

Debit Card / ATM Card

A debit card enables the holder to have his purchases directly charged from funds in his account at a deposit taking bank. ATM card is payment card issued to a person for activating automated teller machine-computer based terminal which allows consumers to transfer and withdraw funds. The account holder is also allotted a pin code to execute the transaction. Now debit cards and ATM cards have almost the same features.

Credit Card (Visa and Master Card)

A card indicating that the holder has been granted a line of credit. I enables the holder to make purchases and / or withdraw cash up to a prearranged ceiling. The credit granted can be settled in full by the end o a specified period or can be settled in parts, with the balance taken as extended credit. Interest is charged on the amount of any extended credi and the holder is sometimes charged an annual fee. Visa card is issued to every client where as Master card, which has normally attractive cred lines is issued to valued clients having highly net worth. Visa and Maste are two different companies through which the bank arranges the issuance of the credit cards.

ATM (Automated Teller Machine)

ATM is a machine installed by the bank to dispense cash to its account holders. It is a computerized machine linked with database of the bank enabling the customer to draw down cash round the clock through magnetically encoded bank's ATM card by using specific PIN code allotted by the bank to the customer. The ATM may be used by the customer for withdrawal, transfer of funds, balance inquiry and mini statement of account. ATM may be operated either on line with real time access to an authorized data base or off line.

Moreover, other multiple banking services are also being provided by the latest ATM machines having diversified functions.

4.2 EXCHANGE OF CURRENCIES

The word foreign exchange is related to the exchange method through which payment in connection with international trades are made. It covers the method by which the currency of a country is exchanged for that of another.

Different countries use different forms of currency and their units of money are called by various names. The United Kingdom uses the Sterling Pound, the United State of America uses the American Dollar, Thailand uses the Baht, Malaysia uses the Ringgit, Indonesia uses the Rupiah, the Philippine uses the Peso and Singapore uses the Singapore Dollar. We can buy or sell foreign currencies at any bank or through a money changer.

Every day, major banks display the exchange rate of the various currencies. These rates fluctuate every day and are determined by the supply and demand of various currencies.

Table below shows the exchange rate of the various currencies displayed by a private bank.

Currency	Selling TT&OD	Buying TT Clean	Buying O/D T/Cheques
U.S. DOLLAR	84.100	83.800	83.5796
POUND	129.7968	129.4542	129.1092
CANADIAN DOLLAR	83.9412	83.7246	83.5781
AUSTRALIAN DOLLAR	77.7588	77.5820	77.4379
YEN	0.895236	0.893009	0.891447
DANISH KRONE	15.1066	15.0598	15.0334
SAUDI RIYAL	22.4000	22.3449	22.3058
SWISS FRANK	78.3363	78.1133	77.9766
DIRHAM	22.8702	22.8145	22.7746
HKD	10.8193	10.7934	10.7745
SINGAPOREAN DOLLAR	61.4529	61.2484	61.1412
SWEDISH KRONE	11.7391	11.6948	11.6743
BATH	2.6063	2.5976	2.5931
EURO	112.4088	112.0993	111.8057

TT means Telegraphic Transfer.
OD means Over Draft.
TC means Traveller Cheque.

Use above table to solve following examples.

EXAMPLE-1

Convert the value of a given amount of the currency of one country in terms of another currency:

(i) 5,00,000 Pk rupees to Euro
(ii) 50 Pounds to US Dollars.
(iii) 250 US dollars to Sterling Pound.
(iv) 5000 Saudi Riyal to Pk rupees.

SOLUTION: (i) Amount = Rs. 5,00,000

Rate of Euro = Rs.112.4088

$$\text{Number of Euros} = \frac{500,000}{112.4088}$$

$$= 4,448.05$$

(ii) Rate of UK Pounds (buying) = Rs.129.4542

Amount of 50 Pounds = $Rs.50 \times 129.4542$

= Rs.6,472.71

Rate of US dollar (selling) = Rs.84.100

Therefore, dollars to get in exchange of 50 pounds = $US\$\frac{6,472.6}{84.100}$

= US$ 76.9644

(iii) Rate of US$ (buying) = Rs.83.800

Therefore 250 US$ = $Rs.250 \times 83.800$ = Rs.20,950

Rate of pound (selling) = Rs.129.7968

Therefore, pounds to be obtained for 250 dollars = $\frac{20950}{129.7968}$

= 161.406 pounds

(iv) Rate of Saudi Riyal = Rs.22.3449

Amount in Pk rupees = $5,000 \times 22.3449$

Rate of pound (selling) = Rs.129.7968

= Rs.1,11,724.5

EXAMPLE-2

A Pakistani employee in Saudi Arabia earns 3200 Riyals a month. He spends 2500 Riyals a month. Determine:

(i) His monthly saving in rupees if 1 Riyal = Rs.22.400

(ii) He remits his saving to Pakistan after a year. When he remits rate changes to 1 Riyal = Rs.22.6203, determine increase in saving due to change in rate.

SOLUTION: Earning per month = 3200 Riyals

Expenses per month = 2500 Riyals

Saving = 3200 − 2500 = 700 Riyals

Saving in Rupees

1 Riyal = Rs.22.400

700 Riyal = 700×22.400

= Rs.15,680

Increasing in saving due to change in rate

Saving for a year $= 700\times 12 = 8400$ *Riyal*

Saving per year at previous rate

$= 8400\times 22.400$

$= Rs. 1,88,160$

Saving per year in rupees with increase in rate

$= 8400\times 22.6203 = 1,90,010$

increase in saving $= 1,90,011 - 1,88,160$

$= Rs. 1850$

EXERCISE – 4.1

1- Convert *250* US Dollar into Sterling Pound.

2- Convert *5000* Riyals into Pak rupees.

3- An importer imports a car from Japan for *5000 yen*. Delivery was to be made after three months. At the time of contract *1 yen = Rs 0.895236*. At the time of delivery *1 yen = Rs 0.892236*. Payment was made at the time of contract. Determine the profit or loss of the importer.

4- A customer wants to convert *150* American dollers into rupees. He goes; to an authorised dealer. He offers him conversion at the rate of *1* dollar = *Rs.84.100*. If it is converted with a money changer, the rate is *1* dollers = *Rs.83.4495*, determine the amount into rupees if it is converted with:

(i) Authorised dealer

(ii) Money Changer

(iii) The loss due to conversion with the money changer.

5- Rate of tea in Pakistan is *Rs.2.1* per pound. Find the rate per Kilogram.

(i) *1* pound = *0.4536* kilogram

(ii) What will be the rate in Saudi Arabia if Saudi Riyal = *Rs.22.400*.

6- An exporter of carpets exports to England Carpets amounting to *40000* Sterling Pound. The spot buying rate exchange at that time was *Rs.129.4542* to *1* Sterling. He receives the amount at the time when rate is *Rs.129.0599* to *1* Sterling. How much he looses?

7- A Pakistani living in Saudi Arabia earns *4370 Riyals* a month. His monthly expenses comes to *3450 Riyals*. He remits his saving monthly to Pakistan. How much he saved in a year if rate of exchange is *Rs.22.400 = 1 Saudi Riyal*. After a year Rate of exchanges is *Rs.22.3004*. Determine the loss due to monthly remitance.

8- Rizwan purchases a car in Saudi Arabia for *15000* Riyals. Delivery was to be made after three months and payment is also to be made at the time of delivery.
At the time of contract, the rate was *1 Riyal = Rs.22.400*, while at the time of delivery the rate was *1 Riyal = Rs.22.0827*. Determine the profit or loss in rupees due to change in the rate.

9- A friend of Ali living in Saudi Arabia remits Ali *450* Riyals. The bank offers two conversions rate.
T.T. Buying *Rs.22.3449 = 1 Riyal*
T/C Buying Rate: *Rs.22.2146 = 1 Riyal*
Which one of the rate will be profitable and also calculate the amount in rupees.

4.3 PROFIT/MARKUP

With the advent of Islamic Banking in Pakistan, the word "interest" has been replaced with the word "Profit or Markup" under Profit and Loss sharing and Islamic Modes of Financing.

When we deposit surplus funds into a bank, we receive some return for allowing the bank to use our money. This return which is given by the bank to us in exchange of using our funds is called **profit on deposit**. The profit so paid is cost or expense for the bank but it is income for the depositor or account holder.

Similarly, when we borrow funds from the bank, we have to pay a some extra amount for using the borrowed money. The extra amount which we pay to bank for using the borrowed money is called **markup**. This is income for the bank but expenses for the borrower or client.

The **profit on deposit** and **markup on loan** is calculated as a percentage which is called the **rate.** The sum which is deposited/invested or borrowed is called **principal.** Profit or markup is usually calculated at a fixed yearly rate called **rate per annum**. Sometimes interest rate is calculated on half yearly or quarterly, monthly or even on daily basis. The amount of profit or markup depends on the length of time the money deposited or borrowed. If profit or markup is calculated on the original principal, it is called simple **profit or markup.** When the profit or markup is added to the principal, the sum is called the **amount.** If the profit or markup is calculated on this amount for the next year, the profit/markup will be called **compound,** which means profit on profit along with principal amount.

The major terminology may be defined as under;

1- The amount/capital borrowed or lent is called **principal.**

2- The percentage of profit charged is called **rate.**

3- The period of the loan or deposit is called the **time** or **period.**

4- When the profit is added to the principal, the sum is called the **amount.**

4.3.1 Calculation of Profit

Profit is the amount which is paid by the bank on the deposits maintained by the clients with the bank. The deposit rate depends upon the period of the deposit. The mode of payment of profit may be monthly, quarterly, half yearly and yearly. The rate or percentage of the principal amount for a period of time usually one year. The rate of profit are declared by the bank on six month basis or on annual basis. Under conventional banking, interest rate on deposit is on predetermined basis. In Islamic banking, there is no concept of pre-determined profit rate, instead the depositor share in the profit and loss of the bank. The profit on deposit is cost of the bank and is treated as expenses of the bank.

Following formula may be used to calculate profit on deposit or sum invested in any bank, financial institutions or National Saving Centers.

$$\text{Profit (Simple)} = \frac{Principal \times Time \times Rate}{100}$$

By the above formula, the Profit is calculated on annual basis. Profit on daily, monthly, quarterly, half yearly basis may be determined by dividing the relevant figure of period (days or months) as per requirements. Moreover by re-arranging the above equation, we may determine any missing component if remaining information are available.

$$\text{Principal} = \frac{Amount\ of\ Profit \times 100}{Rate \times Time}$$

(In case profit, time and rate are given)

$$\text{Principal} = \frac{Total\ Amount \times 100}{100 + (Time \times Rate)}$$

(In case amount, time and rate are given)

$$\text{Rate} = \frac{Amount\ of\ Profit \times 100}{Time \times Principal}$$

$$\text{Time/Period} = \frac{Amount\ of\ Profit \times 100}{Rate \times Principal}$$

(In case profit, rate and principal are given)

The compound profit (*profit on profit*) may be determined with the help of following formula

$$\text{Final Amount} + Principal\left(1 = \frac{Rate}{100}\right)^{Time}$$

$$\text{Compound Profit} = \text{Final Amount} - \text{Principal}$$

EXAMPLE-1

If we invest Rs. 1000 in a saving account @ 10 % profit per year. How much would we have in one year?

SOLUTION:

$$Profit = Rs.1000 \times \frac{10}{100} = Rs.100$$

$$\text{Total investment} = Rs.1000 + Rs.100 = Rs.1100$$

Thus Rs.1100 is equal to the original principal of Rs.1000 plus 10 % p.a. We may say that Rs.1100 is the future value of Rs.1000 invested for one year @ 10 % profit rate.

EXAMPLE-2

Ali deposited Rs.2000 in National Bank of Pakistan (NBP) in his saving A/C for 2 years @ 5 % p.a. What would be the amount of profit on deposit for 2 years to be paid by NBP.

SOLUTION:

$$P = Rs.2000$$

$$R = 5\ \%$$

$$T = 2 \text{ years}$$

$$\text{Profit for 1st year} = \frac{P \times R}{100}$$

$$= Rs.\frac{2000 \times 5}{100} = Rs.100$$

$$\text{Principal amount} = Rs.2000 + Rs.100 = Rs.2100$$

$$\text{Profit for 2nd year} = Rs.\frac{2100 \times 5}{100} = Rs.105$$

$$\text{Total profit paid for two years} = Rs.100 + Rs.105 = Rs.205$$

EXAMPLE-3

If the simple profit on Rs.640 for 12 years is Rs.384 find the rate of profit.

SOLUTION: Principal $= Rs.640$

Simple profit $= Rs.384$

Time $= 12\ years$

$$Rate = \frac{Amount\ of\ Profit \times 100}{Time \times Principal}$$

$$= \frac{384 \times 100}{640 \times 12} = 5\%$$

EXAMPLE-4

How much time a sum of Rs.9400 will take to become Rs.10,951, if the same is invested @ $3\frac{2}{3}\%$ p.a.

SOLUTION: Principal $= Rs.9400$

Amount $= Rs.10,951$

Simple profit $= Rs.10951 - Rs.9400 = Rs.1551.$

$$Rate = 3\frac{2}{3}\% = \frac{11}{3}\%$$

$$Period\ /\ time = \frac{Amount\ of\ Profit \times 100}{Rate \times Principal}$$

$$= \frac{1551 \times 100 \times 3}{9400 \times 11} = \frac{9}{2}\ years = 4\frac{1}{2}\ years.$$

EXAMPLE-5

Find the compound profit on Rs.4000 at 5% for 3 years.

SOLUTION: Principal $= Rs.4000$

Rate $= 5\%$

Time $= 3\ years$

$$Final\ amount = (4000) \times \left(1 + \frac{5}{100}\right)^3$$

$$= 4000 \times \frac{105}{100} \times \frac{105}{100} \times \frac{105}{100} = \frac{9261}{2} = Rs.4630.50$$

Compound profit $= Rs.4630.50 - Rs.4000 = Rs.630.50.$

EXAMPLE-6

Find the compound profit of Rs.5000 at 4% p.a. for 2 years and 6 months.

SOLUTION:

$$\text{Principal} = Rs.5000$$

$$\text{Rate} = 4\%$$

$$\text{Time} = 2\frac{1}{2}\text{ years}$$

$$\text{Final amount} = 5000\times\left(1+\frac{4}{100}\right)^2\left(1+\frac{2}{100}\right)$$

$$= 5000\times\frac{104}{100}\times\frac{104}{100}\times\frac{102}{100}=\frac{137904}{25}=Rs.5516.16$$

$$\text{Compound profit} = Rs.5516.160 - Rs.5000 = Rs.516.16.$$

EXAMPLE-7

Find the compound profit on Rs.1500 for 2 years at 6%, annually payable half yearly.

SOLUTION:

$$\text{Principal} = Rs.1500$$

$$\text{Rate} = 6\%\ p.a = 3\% \text{ for half yearly}$$

$$\text{Time} = 2 \text{ years or } 4 \text{ half yearly}$$

$$\text{Final amount} = 1500\times\left(1+\frac{3}{100}\right)^4$$

$$\text{Final amount} = 1500\times\frac{103}{100}\times\frac{103}{100}\times\frac{103}{100}\times\frac{103}{100}=Rs.1688.26$$

$$\text{Compound profit} = Rs.1688.26 - Rs.1500 = Rs.188.26.$$

Calculation of Markup

Income earned from charging markup upon funds advanced by a bank. Under Islamic mode of financing, the interest earned by the bank is named as markup. Markup rate is fixed by the bank keeping in view the bench- marked rate issued by the State Bank of Pakistan on daily basis. This bench marked rate is called KIBOR- Karachi Inter Bank Offer Rate, issued by SBP for one month, 3 months , 6 months, 9 months and 12 months. Banks charge the markup rate by adding their cushion to the KIBOR rate. This cushion is normally called the spread, which is difference between deposit rate and lending rate. Markup is the main source of income for the commercial bank. It may be earned, realized from the clients and credited to the banks income account. In Islamic banking, there is no pre- determined mark up rate and financing is done on the basis of sale and buy back agreement for goods and bill under which the sale price is determined by adding a markup on the purchase price.

The markup on loan or borrowed money may be calculated with the help of following formulas.

$$\text{Markup per annum} = \frac{\text{Amount Borrowed} \times \text{Rate}}{100}$$

$$\text{Markup per month} = \frac{\text{Amount Borrowed} \times \text{Rate}}{100 \times 12}$$

$$\text{Markup per day} = \frac{\text{Amount Borrowed} \times \text{Rate}}{100 \times 365}$$

EXAMPLE

A man borrows Rs.100,000 for 3 year at rate of 16 % p.a. What is the markup he has to pay?

SOLUTION: The principal amount $= \text{Rs.}100,000$

The markup of Rs.100,000 for 1 year $= \frac{16}{100} \times 100,000$

$= \text{Rs.}16000$

The markup of Rs.100,000 for 3 years $= 3 \times 16000$

$= \text{Rs.}48,000$

EXERCISE – 4.2

1- A financial institution charges *Rs.55* simple profit on a sum of money which is borrowed for five months. Given that the rate of profit is *12%* per annum, find the sum of money.

2- Mrs.Javed invests in Savings Scheme *Rs.800* at *6%* per annum and *Rs.1,200* at *7%* per annum. What is her total amount of profit on these two investments?

3- How long would *Rs.1250* have to be deposited at *6%* per year simple profit to gain *Rs.750* simple profit?

4- Ali lent to Abid *Rs.4,800* for *7* months. At the end of this period Abid had to pay Ali profit of *Rs.119*. What was the rate of simple profit per annum?

5- In a certain year, Javed puts *Rs.600* in a private bank at the end of March and *Rs.400* in the same bank at the end of June. The bank offers *3%* per annum simple profit rate. Find the total amount Javed receives from the bank at the end of December in that year?

6- At what annual rate of profit would a sum of *Rs.680* will increase to *Rs.850* in *3* years and *4* months?

7- Copy and complete the following table with the help of formula given in this unit?

	principal	Profit rate	Time	Simple profit	Amount
(a)	Rs. 12,000	8%	7 years		
(b)	Rs. 500	11%		Rs. 220	
(c)		9%	4 years	Rs. 108	
(d)	Rs. 3,000		10 years	Rs. 1,200	
(e)			2 years	Rs. 360	Rs. 3,960
(f)	Rs. 1,800		18 years	Rs. 189	
(g)	Rs. 4,500		2 years		Rs. 5,040
(h)		5%		Rs. 90	Rs. 1,290

8- A bank increased the rate of profit from *3.5%* to *4%* per annum. Find how much more profit Saeed would receive if he deposited *Rs.6400* in the bank for *6* months at the new profit rate.

9- Mrs. Jamshed invested *Rs.4000* in XYZ Bank Limited which paid simple profit at a rate $7\frac{1}{4}\%$ per annum to its investors. After *2* years, the rate was increased to *8%* per annum. Find the amount she had at the end of *7* years.

10- Mr.Dawood deposits a certain sum of money in ABC Limited. If the profit rate of the bank decreases from $3\frac{3}{4}\%$ per annum to $3\frac{1}{2}\%$ per annum, Mr. Dawood's profit will decrease by *Rs.50* in a year. Find the sum of money he deposits.

11- Find the compound profit on.

(i) *Rs.450* for *2* years at *10%* per annum compounded yearly;

(ii) *Rs.700* for *3* years at *11%* per annum compounded yearly;

(iii) *Rs.5000* for *2* years at $11\frac{3}{4}\%$ per annum compounded yearly;

(iv) *Rs.1200* for *3* years at *4%* per annum compounded yearly;

(v) *Rs.10000* for *3* years at $7\frac{1}{2}\%$ per annum compounded yearly;

12- Waseem invests *Rs.5000* at $5\frac{1}{4}\%$ per annum profit compounded annually. Find the amount at the end of the third year.

13- Javed invests *Rs.800* at $12\frac{1}{2}\%$ per annum compound profit compounded half-yearly. What is the amount at the end of the first year?

14- Mr.saleem invests *Rs.9000* at *2%* per annum compound profit compounded daily. What is his amount at the end of the third day.

Solve Problems Related to Commercial Banking and National Saving Schemes.

EXAMPLE-1

What sum would borrow in Rs.174 as markup at 5% in 4 years.

SOLUTION:

Markup = Rs.174

Time = 4 years

Rate = 5% P.A

$$\text{Principal} = \frac{174 \times 100}{5 \times 4} = Rs.870$$

EXAMPLE-2

Find the markup on amount of Rs.6900 borrowed on 18th of May and repaid on 11th of October of the same year at $3\frac{1}{2}$ percent p.a.

SOLUTION:

Principal amount = Rs.6900

Rate = $3\frac{1}{2}\%$

Time = 146 days or $\frac{146}{365}$ years

(From 18th May to 11th October)

May	13 Days
June	30 Days
July	31 Days
August	31 Days
September	30 Days
October	11 Days
Total	146 Days

$$\text{Markup} = \frac{6900 \times 146 \times 7}{365 \times 2 \times 100} = Rs.96.60$$

Amount = Rs.6900 + Rs.96.60 = Rs.6996.60

EXAMPLE-3

What sum of money would produce Rs.630.50 in 3 years at 5% compound profit.

SOLUTION: Suppose sum = Rs.100

Rate = 5%

Time = 3 years

$$\text{Final amount} = 100\times\left(1+\frac{5}{100}\right)^3$$

$$= 100\times\frac{105}{100}\times\frac{105}{100}\times\frac{105}{100} = Rs.\frac{9261}{80}$$

$$\text{Compound profit} = \frac{9261}{80}-100 = \frac{1261}{80}$$

$$\text{Given compound profit} = 630.50 = \frac{1261}{2}$$

If compound profit is $Rs.\frac{1261}{80}$ then principal = Rs.100

If compound profit is Re.1 then principal $= \frac{100\times 80}{1261}$

If compound profit is $Rs.\frac{1261}{2}$ then principal $= \frac{100\times 80\times 1261}{1261\times 2}$

$= Rs.4000.$

EXAMPLE-4

What is the difference between simple and compound profit on Rs.25000 for 4 years at 5% p.a.

SOLUTION: Principal = Rs.25000

Time = 4 years

Rate = 5%

$$\text{Simple profit} = \frac{25000\times 4\times 5}{100} = Rs.5000.$$

$$\text{Final amount} = 25000\times\left(1+\frac{5}{100}\right)^4$$

$$= 25000\times\frac{105}{100}\times\frac{105}{100}\times\frac{105}{100}\times\frac{105}{100} = Rs.30,387.65$$

Compound profit = Rs.30387.65 − Rs.25000 = Rs.5387.65

Difference between simple profit and compound profit

= Rs.5387.65 − Rs.5000 = Rs.387.65

EXERCISE 4.3

1- A man borrowed *Rs.1460* from ABC Bank on the *3rd* of March at $12\frac{1}{2}$ % annually. What should he pay on the *1st* of July to pay off the debt.

2- A shopkeeper borrowed *Rs.3540* from ABC Bank at $10\frac{3}{4}$ % and lent the whole amount at $11\frac{1}{2}$ % on the same day, what would be gained from this after *3* years and *4* months.

3- XYZ Bank gained *Rs.8034* on its loan at *6%* compound markup in *2* years. What amount did it lend?

4- A Company borrowed *Rs.6,600* from ABC Bank Ltd at *8%* simple markup per annum. How much did the company owe to the bank at the end of *11* months?

5- XYZ Bank charges *2.25%* per month simple markup on personal loans. If Ali borrows *Rs.6,400* for a period of *2* years *1* month, find the total markup he has to pay to XYZ Bank.

6- Find out the compound markup on *Rs.250,000* for one year @ *14 %* compounded markup annually.

7- Find compound profit on *Rs.600* for *4* years at *6* percent per annum.

8- Find the compound profit of *Rs.50000* at *4%* for $1\frac{1}{2}$ years.

9- Find the compound profit on *Rs.54000* for one year at *12%* per annum.

4.4 INSURANCE

4.4.1 Insurance

Insurance is important tool of risk management in the business transactions or business dealing. Insurance is a contract between two parties whereby a person or a party agrees to pay an amount in monthly/quarterly or yearly installment, to a certain insurance company, in order to cover / indemnify the risks associated with life, theft, damages etc for which contract of insurance is made. Under this contract, the insurance company has to pay back the agreed amount or the actual amount of loss or damages etc. on sudden death, danger or maturity.

The key terms used in insurance are:

(i) An insurance company or an organization who insures, provides insurance cover against various risks and issues insurance policies is called **insurer.**

(ii) A person to whom an insurance policy is issued; the beneficiary in a contract of insurance is called **insured or insurant.**

(iii) The contract which is executed between two parties is called **insurance policy.**

(iv) The periodic installment to be paid by the insured is called **premium.**

(v) The time period agreed upon by both the parties (insured and insurer) is called **maturity**.

(vi) The agreed amount to be paid back on maturity or expiry of the agreed period, includes the actual amount paid in installments plus profit is termed as **bonus**.

4.4.2 Life Insurance and Vehicle Insurance

In general, there are two types of insurance:

1- Life Insurance.

2- Vehicle and Property Insurance.

Life Insurance

Life insurance is a contract wherein a maturity period is agreed between the parties to pay back a sum equal to original amount and the profit, which is called bonus. Otherwise it is paid on death or in case of accident etc, whichever comes earlier. A person can also get an insurance policy against old age or any disability, the amount of which may range from *10%* to *50%* of the income of the insurer. Here is an example for calculating the amount for yearly, half yearly, quarterly or monthly premium for life insurance.

For example:

The age of an insured is 30 years at the time of insurance.

Rate for annual premium is Rs. 4.5 % of the total amount of policy.

The rate for half yearly premium is 52 % of the annual premium.

The rate for quarterly premium is 27 % of the annual premium.

The rate for monthly premium is 9 % of the annual premium.

Total amount of policy = Rs. 4,00,000

$$\text{1st premium @ } 4.5\% = \frac{4.5}{100} \times 4{,}00{,}000 = Rs.18{,}000$$

$$\text{Policy fee @ } .25\% = \frac{.25}{100} \times 4{,}00{,}000 = Rs.1000$$

$$\text{Annual premium} = \text{1st premium} + \text{policy fee}$$

$$= 18000 + 1000 = Rs.19000$$

Policy fee is charged *Rs.1000* or *0.25%* of the purchasing amount of policy. If *0.25%* of the policy amount exceeds *Rs.1000*, then only *Rs.1000* will be charged as policy fee.

Now, 1st premium = Rs.18000.

$$\text{Half yearly premium} = \frac{52}{100} \times (\text{1st premium} + \text{policy fee})$$

$$= \frac{52}{100} \times (18000 + 1000)$$

$$= \frac{52}{100} \times 19000$$

$$= 52 \times 190 = Rs.9880$$

$$\text{Quarterly premium} = \frac{27}{100} \times 19000$$

$$= Rs.27 \times 190$$

$$= Rs.5130$$

$$\text{Monthly premium} = \frac{9}{100} \times 19000$$

$$= Rs.9 \times 190$$

$$= Rs.1710$$

It is important to note that the amount of policy premium and the time of maturity are fixed in accordance with the age of the insured as per rules of the company. Usually as the age of the insured increase the maturity period decreases. In other words, the higher the age of the insured, the lower would be the maturity period.

EXAMPLE

Calculate the first, quarterly and monthly premium if the age of the insured is 30 years, policy amount is Rs.3,00,000, maturity time 25 years, rate of premium 3.5% fixed with policy fee @ 0.25%.

SOLUTION:

Let the age of an insured at the time of insurance = 30 years.

Policy amount = Rs.3,00,000

Maturity time = 25 years

Premium is fixed @ 3.5 %

Policy fee @ .25 %

$$\text{1st premium} = \frac{3.5}{100} \times 3{,}00{,}000$$

$$= 35 \times 300$$

$$= Rs.10{,}500$$

$$\text{Policy fee @ .25\%} = \frac{0.25}{100} \times 3,00,000$$
$$= 25 \times 30$$
$$= Rs.750$$

$$\text{Family income contract @ 0.5\%} = \frac{0.5}{100} \times 3,00,000$$
$$= Rs.1500$$

$$\text{Total amount paid} = 1st \text{ premium} + \text{policy fee} + \text{family income contract}$$
$$= Rs.10500 + Rs.750 + Rs.1500$$
$$= Rs.12750$$

When the policy matures:

$$\text{Policy amount} = Rs.3,00,000 \text{ ....}(i)$$

$$\text{Bonus @ 4.5\% for 25 years} = Rs.\frac{4.5}{100} \times 3,00,000 \times 25$$
$$= 45 \times 300 \times 25$$
$$= 1125 \times 300$$
$$= Rs.3,37,500 \text{ ....}(ii)$$

$$\text{Maturity bonus @ 1.5\% for 20 years} = Rs.\frac{1.5}{100} \times 3,00,000 \times 20$$
$$= Rs.15 \times 6000$$
$$= Rs.90,000 \text{ ....}(iii)$$

$$\text{Terminal bonus @ 1.6\%} = \frac{1.6}{100} \times 3,00,000 \times 20$$
$$= 16 \times 6000$$
$$= Rs.96000 \text{ ....}(iv)$$

Family income bonus @ 0.75 % $= \frac{0.75}{100} \times 3{,}00{,}000 \times 20$

$= Rs.600 \times 75$

$= Rs.45{,}000$ ....(v)

Total money he will get $= \boxed{i} + \boxed{ii} + \boxed{iii} + \boxed{iv} + \boxed{v}$

$= Rs.(3{,}00{,}000 + 3{,}37{,}500 + 90{,}000 + 96000 + 45{,}000)$

$= \boxed{Rs.8{,}68{,}500}$

Premium paid = *total amount paid × maturity time*

$= Rs.12750 \times 25$

$= \boxed{Rs.3{,}18{,}750}$

> In case of death of the insured within one year after getting the policy, his family gets the following due to family income contract.

Policy amount = Rs. 3,00,000

Bonus @ 4.2 % $= \frac{4.2}{100} \times 3{,}00{,}000$

$= 42 \times 300$

$= Rs.\ 12{,}600$

Total sum = Rs.300,000 + Rs.12,600 = Rs. 312,600

> In addition to the above amount the family will get Rs. 30,000 @ 10 % of policy amount yearly as their income for 24 years. Thus the total amount the insured family will get is:

Rs. 3,12,600 + Rs. 7,20,000

$= \boxed{Rs.10{,}32{,}600}$

Vehicle Insurance

Sometimes the person or companies get insurance policies against their vehicles or properties to cover the risk of theft, accidents, fire etc. The amount of the insurance is the total price or the partial price of the object and the premium in some percentage of the actual price of the object or the total amount of the policy and it is decided accordingly as per rules of the company at different rates for different time periods. The first premium is usually the total amount of one year installments.

EXAMPLE-1

A person got an insurance policy for his car at the rate of 3.5%. He paid an amount of Rs.14500 as the 1st premium of one year. How much is the price of his car while he had paid Rs.500 as service charges?

SOLUTION: Let the total price of the car = Rs. x.

Total amount he paid = Rs. 14500

Amount paid as service charges = Rs. 500

Remaining amount of premium = Rs. 14500 − 500

= Rs.14000

Now $3.5\ \%\ \text{of}\ x = \text{Rs. } 14000$

$$\frac{3.5}{100} \times x = \text{Rs.} 14000$$

$$x = \text{Rs.} \frac{14000 \times 100}{3.5}$$

$$x = \frac{14000 \times 100 \times 10}{35}$$

$$= 400 \times 100 \times 10$$

$$= 400 \times 1000$$

$$= \text{Rs. } 4{,}00{,}000$$

Price of the car = Rs. 4,00,000

In vehicle insurance the yearly premium reduces as the value of the assets depreciates and is calculated according to the depreciated price.

Usually some service charges are also included in the premium but we are discussing the cases without service charges etc.

EXAMPLE-2

A person got an insurance policy for his car at the rate of 3.6%. He paid an amount of Rs.12206 as the 1st premium of one year. How much is the price of his car while he had paid Rs.200 as service charges?

SOLUTION: *Let the total price of his car be x rupees.*

$$\text{Total amount paid} = Rs.12206$$

$$\text{Paid as service charges} = Rs.200$$

$$\text{Remaining amount of Premium} = Rs.12206 - 200$$

$$= Rs.12006$$

$$\text{Now } 36\text{ \% of } x = Rs.12006$$

$$\frac{3.6}{100}x = Rs.12006$$

$$x = Rs.\frac{12006 \times 100}{3.6}$$

$$= Rs.\frac{12006 \times 100 \times 10}{36}$$

$$\text{Price of car} = x = Rs.333,500$$

In vehicle or property insurance the yearly premium reduces as the value of the assets depreciates and is calculated according to the depreciated price.

In case of fire, loss etc. the reimbursement of the policy is made according to the current value of the assets.

EXAMPLE-3

Find the total amount of insurance Arslan has to pay for his car for a period of 5 years if the value of the car is Rs.8,50,000, rate of insurance is 4.5%. The insurance is to be paid yearly and the last year's insurance is 0% while the depreciation is @ 10% per year.

SOLUTION:

$$\text{Value of the car} = Rs.850,000$$

$$\text{Rate for insurance} = 4.50\%$$

$$\text{Tenure} = 5\text{ years}$$

$$\text{1st year's insurance} = Rs.\frac{4.5}{100} \times 850,000 = Rs.38250$$

$$\text{Depreciation} = \frac{10}{100} \times 8,50,000 = Rs.85000$$

$$\text{Depreciated price} = Rs.850,000 - 85000$$

$$= Rs.765000$$

$$\text{2nd year's insurance} = Rs.\frac{4.5}{100}\times 7,65,000$$
$$= Rs.34425$$
$$\text{Depreciation} = Rs.\frac{10}{100}\times 7,65,000$$
$$= Rs.76,500$$
$$\text{Depreciated price} = Rs.7,65,000 - 76,500$$
$$= Rs.6,88,500$$
$$\text{3rd year's insurance} = Rs.\frac{4.5}{100}\times 6,88,500$$
$$= Rs.30,982.50$$
$$\text{Depreciation} = Rs.\frac{10}{100}\times 6,88,500$$
$$= Rs.68,850$$
$$\text{Depreciated price} = Rs.6,88,500 - 68,850$$
$$= Rs.6,19,650$$
$$\text{4th year's insurance} = Rs.\frac{4.5}{100}\times 6,19,650$$
$$= Rs.27,884.25$$
$$\text{5th year's insurance} = 0$$

So the total amount paid as insurance will be

1st year	Rs.	38,250
2nd year	Rs. +	34,425
3rd year	Rs. +	30,982
4th year	Rs. +	27,884
5th year	Rs. +	0
	Rs.	1,31,541

4.4.3 Simple Real Life Problems Regarding Purchase of Life and Motor Vehicle Insurance.

Let us consider the following examples showing the importance of life and motor vehicle insurance.

EXAMPLE-1

For an insured whose age is 30 years at his nearest birthday. The rate for annual premium is 4.842% of the total amount of policy. The rate for half yearly installments is 52% of the annual premium. The rate for quarterly premium is 27% of the annual premium. The rate for monthly premium is 9% of the annual premium.

SOLUTION:

The total amount of policy $= Rs.100,000$

1st premium @ 4.842% $= Rs.\frac{4.842}{100}\times 100,000 = Rs.4842$

Policy fee @ 0.25% $= Rs.\frac{.25}{100}\times 100,000 = Rs.250$

Annual premium $= Rs.5,092$

Some times the amount of policy fee is restricted, it may be at the most Rs.200 which means that if 0.25% of the amount increases Rs.200, even then only Rs.200 will be charged. Let us suppose that fee is restricted to Rs.200 instead of Rs.250.

So in above case 1st premium may be $4842 + 200 = 5,042$.

Half yearly premium $= Rs.\frac{52}{100}\times 5,042 = Rs.2,621.84$

$\simeq Rs.2,622$

Quarterly premium $= Rs.\frac{27}{100}\times 5,042 = Rs.1,361.34$

$\simeq Rs.1,361$

Monthly premium $= Rs.\frac{9}{100}\times 5,042 = Rs.453.78$

$\simeq Rs.454$

Let us now consider the example showing the importance of life insurance policies or otherwise when his policy matures he will get the following amount.

Policy amount $= Rs.100,000$

Bonus @ 4.2% for 25 years $= Rs.\frac{4.2}{100}\times 100,000 \times 25$

$= Rs.4,200 \times 25$

$= Rs.1,05,000$

Maturity Bonus @ 1.4 % for 20 years $= Rs.\frac{1.4}{100}\times 100,000\times 20$

$= Rs.28,000$

Terminal Bonus @ 1.5 % $= Rs.\frac{1.5}{100}\times 100,000\times 20$

$= Rs.30,000$

Family income Bonus $= Rs.\frac{0.75}{100}\times 100,000\times 20$

$= Rs.15,000$

Family income Bonus $= Rs.(100000 + 105000 + 28000 + 30000 + 15000)$

$= Rs.278,000$

While he had paid only as premium $= 4537\times 25$

$= Rs.113425$

EXAMPLE-2

The price of a car is Rs. 12,50,000 whereas the rate for premium is Rs. 4.50 % for a tenure of 5 years. Calculate the total amount paid as insurance, while depreciation is 10 % yearly.

SOLUTION:

Price of car = Rs. 12,50,000

Rate of premium = 4.5 %

Tenure = 5 years

Premium is to be paid on yearly basis,

while depreciation is @ 10 % yearly.

1st premium $= \frac{4.5}{100}\times 12,50,000 \doteq \boxed{Rs.56250}$

Depreciation $= \frac{10}{100}\times 12,50,000 = Rs.1,25,000$

Depreciated price $= 12,50,000-1,25,000$

$= Rs.11,25,000$

$$\text{2nd premium} = \frac{4.5}{100} \times 11,25,000 = \boxed{Rs.50,605}$$

$$\text{Depreciation} = \frac{10}{100} \times 11,25,000 = Rs.1,12,500$$

$$\text{Depreciated price} = Rs.11,25,000 - Rs.1,12,500$$

$$= Rs.10,12,500$$

$$\text{3rd premium} = \frac{4.5}{100} \times 10,12,500 = \boxed{Rs.45562.50}$$

$$\text{Depreciation} = \frac{10}{100} \times 10,12,500 = Rs.1,01,250$$

$$\text{Depreciated price} = Rs.10,12,500 - Rs.1,01,250$$

$$= Rs.\ 9,11,250$$

$$\text{4th premium} = \frac{4.5}{100} \times 9,11,250$$

$$= Rs.\ 41,006.25$$

$$\text{5th premium} = 0$$

So total premium paid as insurance will be

1st year	Rs.	56250.00
2nd year	Rs.	50605.00
3rd year	Rs.	45562.50
4th year	Rs.	41006.25
5th year	Rs.	0
	Rs.	193423.75

> At the time of maturity, if no claim has been applied by the insured then some amount already decided by the parties may be paid back to the insured as non-claim bonus.

EXERCISE – 4.4

1- If the amount of premium is calculated as.
Yearly premium @ *4.5%* of the policy *income with policy* fee @*0.25%* of the policy amount or at the most *Rs.200*.
Half yearly premium @ *52%* of yearly premium.
Quarterly premium @ *27%* of yearly premium.
Monthly premium @ *9%* of yearly premium.
Then complete the table below for calculation of the premiums.
Also find the total amount he pays to the company.

Amount of policy	*Yearly premium*	*Half yearly premium*	*Quarterly premium*	*Monthly premium*
(i) 50,000				
(ii) 100,000				
(iii) 150,000				
(iv) 200,000				

2- Calculate the amount to be received by the heirs of an insured if he died *2* years after buying the policy while.
The amount of policy = *Rs.50,000*
Premium is fixed @ *4.2%* yearly
Policy fee @ *0.3%*
Family income contract @ *0.6%*
Maturity period = *22* years
Bonus @ *4.5%* and *Rs.6000* yearly income is promised by the company.

3- Mr. Ahmed Ali insured his house worth *Rs.75,00,000* @*2%* for *4* years calculate the amount paid in *4* years, while the rate of depreciation is *10%* yearly.

4- Mr. Nadeem insured his shop *@3%* for *3* years, the depreciation rate is *5%* yearly. If he paid an amount of *Rs.21000* as the *1st* premium, what is the worth of his shop. If he got a claim of *Rs.200,000* after two years, how much benefit did he get?

5- Mr. Adil bought a running business worth *Rs.10,00,000* and got it insured *@2.5%* as yearly premium for *4* years. After *3* years he got a claim of *Rs.500,000* for actual damages. How much loss had he recovered through insurance?

6- Mr. Javeed bought an insurance policy against his car worth *Rs.8,50,000, @ 4.25%* for *3* years. What total amount will he pay as premium, if he had not claimed and damages during the period? Where depreciation rate is *10%*.

7- Mr. Rehman bought a vehicle worth *Rs.7,50,000*. He got it insured *@3.5%* for *5* years. How much he paid in total for covering the risks, if he had got a claim of damages worth *Rs.100,000* during the period? Where depreciation rate is *10%*.

8- Ms. Maria bought an insurance policy *@3.25%* for her car for *3* years. Her *1st* premium is *Rs.26000*. Tell the price of her car. Also calculate the amount of her *2nd* and *3rd* premium.

4.5 LEASING/FINANCING

4.5.1 Leasing/Financing

Lease is a contract whereby the owner of an asset, the **lessor**, gives the hirer, the **lessee**, the right to use the asset for a specified period in exchange of rental payment. The ownership of the leased asset during the lease period remains with the lessor. Assets such as real estate, machinery equipment or other fixed assets are leased out as per the lease agreement. Hire purchase is also a similar mode of financing which is widely used to financial fixed assets.

A leasing contract for machine and equipment or other fixed assets usually has an element of financing, as the right to use the assets is received by the lessee without paying the cost of the asset and thus leasing arrangement is equivalent to source of finance.

The bank and the leasing companies provide assets to its customer under lease agreement. Their profit is based on the difference between the cost of funds they borrow or invest to acquire the assets and the earning in the shape of rental received from the lessee. The lessor gets the tax benefit for depreciation as the ownership of assets remain with the leasing company and the lessee shows the rental payment as expenses in the income statement. Leasing provides an alternative to purchase an asset in order to acquire its services without directly incurring any fixed debt obligation.

There are two types of lease available to the business firms.

(i) Operational Lease

It is short-term lease which is cancelable at the option of the firm leasing the asset. Such leases are commonly used for leasing such items as computer hardware, vehicles and equipment etc.

(ii) Financial Lease

It is a long term lease which is non-cancelable contractual commitment on the part of the lessee to make a series of payment to the firm that actually own the asset, the lessor, for use of the assets.

4.5.1(i) Leasing/Financing of Motor Vehicle

Auto loan / car financing is a major form of consumer finance. By utilizing the facility a customer can purchase a car by complying with terms of car financing by depositing payment with the bank. The bank covers the shortfall in the finance for the customer against which the bank takes monthly installment from the customer to compensate the bank for the finance provided along with markup charges.

4.5.1 (ii) Down Payment

The customer is required to deposit the down payment with the bank along with the application form. The down payment comprises of *15%* equity value of the car, the insurance costs, one month's installment and the processing fee.

4.5.1 (iii) Motor Vehicle Insurance

Motor vehicle which is financed by the bank is comprehensively insured to cover risks associated with it so that the interest of the bank may be safeguarded adequately. All the vehicles are insured from an approved insurance agency/company of the bank. The bank arranges insurance and the payment of premium which is recovered from the borrower.

4.5.1 (iv) Processing Charges

Processing fee/charges are the amount deducted by the bank to process the request of the client for financing of vehicle. These charges are normally mentioned in schedule of bank charges. These charges may differ from bank to bank. Normally processing fee for the request is ranged from *Rs. 3,000 to Rs. 5,000*.

4.5.1 (v) Repayment in Monthly Installments

It is also called amortization schedule. It is a table which shows the repaying or servicing of a loan/finance amount with periodic payment of principal and interest over the life of the loan.

EXAMPLE-1

Ali leased a car of price Rs.4,50,000 from Leasing Company with an equity deposit Rs.100,000, interest rate 17% for a period of 2 years.

SOLUTION:

On deposit of Rs. 100,000 as an equity Ali can take the car but he is not owner of the car but only the hirer. The ownership of the car will be transferred to him after he has paid of the balance amount of Rs. 350,000 plus the markup on Rs. 350,000 in 2 year at 17 % p.a. by the equal monthly installment.

$$\text{Markup} = \frac{350{,}000 \times 17 \times 2}{100} = \text{Rs.}119{,}000$$

$$\text{Total amount to be paid} = 3{,}50{,}000 + 1{,}19{,}000 = \text{Rs. } 4{,}69{,}000$$

$$\text{Each monthly installment} = \text{Rs.}\frac{4{,}69{,}000}{24} = \text{Rs.}19{,}542$$

EXAMPLE-2

A truck is priced at Rs. 5,00,000. It may be bought at 15 % of down payment or equity. It has to be leased / hired on simple markup of 18 % p.a. for a period of 2 years on monthly installment.

Find the: (i) Monthly installment

(ii) Initial price of truck

(iii) The % age of money saved if the truck is purchased by paying Rs.5,00,000

SOLUTION:

$$\text{Down payment} = \frac{5{,}00{,}000 \times 15}{100} = \text{Rs.}75000$$

$$\text{The saving amount} = 5{,}00{,}000 - 75{,}000 = \text{Rs. } 4{,}25{,}000$$

$$\text{Markup on Rs.}4{,}25{,}000 \text{ for 2 years} = \frac{4{,}25{,}000 \times 18}{100} \times 2 = \text{Rs.}1{,}53{,}000$$

Additional amount to be paid

$$\text{in 24 installments} = 4{,}25{,}000 + 1{,}53{,}000 = \text{Rs. } 5{,}78{,}000$$

$$\text{Monthly installment} = \frac{5{,}78{,}000}{24} = \text{Rs.}24{,}084$$

$$\text{Total amount paid} = \text{down payment} + \text{additional amount}$$

$= Rs.\ 7,5000 + Rs.\ 5,78,000$

$= Rs. 6,53,000$

On financing, the additional amount to be paid $= Rs.\ 6,53,000 - Rs. 5,00,000$

$= Rs. 1,53,000$

Percentage of money that can be saved

on cash term $= \dfrac{1,53,000}{5,00,000} \times 100 = 31\%$

4.5.2 Real Life Problems

EXAMPLE-1

On 1st January, 2001 a machinery is purchased by Ali on the hire purchase system. The payment to be made Rs. 4,000 down (on the signing of contract) and Rs. 4,000 annually for three years. The cash price of the machinery is Rs. 14,900 and the rate of markup is 5%. Calculate the amount of markup and principal to be paid.

SOLUTION:

Dates	Cash Price	Installments	
		markup	principal
	14,900		
Less paid on 1-1-2001 (down payment)	4,000		4,000
	10,900		
Less paid on 31-12-2001	3,455	545	3,455
	7,445		
Less paid on 31-12-2002	3,628	372	3,628
Less paid on 31-12-2003	3,817	183	3,817
		1,100	14,900

Ali will pay Rs.14900 + 1100 = 16000 in total.

Note: (1) The figures are approximated to the nearest rupee.

EXAMPLE-2

M/s Rehman & Co. Ltd, purchased wagons from M/s Haq Engineering Company Ltd. on hire-purcahse system spread over a period of four years. Rs. 12,000 was payable on 1st January 2000 at the date of delivery and the balance by yearly instalments of Rs.12,000 each on 31st December. Haq Engineering charged markup on the yearly balances at the rate of 5% p.a. The cash price of the wagons on delivery was Rs. 54,600. Calculate the amount of markup and principal.

SOLUTION:

Dates	Cash Price	Installments	
		markup	principal
	54,600		
Less paid on 1-1-2000	12,000		12,000
	42,600		
Less paid on 31-12-2000	9,870	2,130	9,870
Less paid on 31-12-2001	32,730	1,637	10,363

EXAMPLE-3

Ahmad purchased a truck on hire-purchase for Rs. 56,000. Payment to be made Rs. 15,000 down and 3 instalments of Rs.15,000 each at the end of each year. Rate of markup is charged at 5% per annum. Calculate the amount of markup and principal separately.

SOLUTION:

period	Cash Price	Installments	
		markup	principal
	56,000		
Less paid on Jan.1st	15,000		15,000
	41,000		
Less paid on Dec.31	12,950	2,050	12,950
	28,050		
Less paid on Dec.31	13,597	1,403	13,597
Less paid on Dec.31	14,453	547	14,453
		4,000	56,000

EXAMPLE-4

M/s Butt Publishing Co. purchased a printing machine on 1st January, 2000. The cash price of machine was Rs. 27,300. The transaction is on hire purchase basis. Rs.6,000 being paid on the singing of the contract and thereafter Rs. 6,000 being paid annually for four years. Markup was charged at 5% p.a. Draw analytical table showing installments (principal + markup). Also determine total amount to be paid by M/S Butt Publishing Co.

SOLUTION: *The amount of principal and interest included in each instalment is worked out as follows.*

Dates	Cash Price	Installments	
		markup	principal
	27,300		
Less paid on 1-1-2000	6,000		6,000
	21,300		
Less paid on 31-12-2000	4,935	1,065	4,935
	16,365		
Less paid on 31-12-2001	5,182	818	5,182
	11,183		
Less paid on 31-12-2002	5,441	559	5,441
Less paid on 31-12-2003	5,742	258	5,742
		2,700	27,300

Total amount to be paid = Rs. 2,700 + Rs. 27,300 = Rs. 30,000

EXAMPLE-5

M.Jahanger Co. Ltd. agreed to purchase a machine on the hire-purchase system for Rs. 4,600. Rs. 600 was paid when the machine was acquired on 1st January, 2001 and the balance was to be paid by annual instalments of Rs. 800 plus markup at 5 per cent.

SOLUTION:

Dates	Cash Price	Installments	
		markup	principal
	4,600		
Less paid on 1-1-2001	600		600
	4,000		
Less paid on 31-12-2001	800	200	800
	3,200		
Less paid on 31-12-2002	800	160	800
	2,400		
Less paid on 31-12-2003	800	120	800
	1,600		
Less paid on 31-12-2004	800	80	800
Less paid on 31-12-2005	800	40	800
		600	4,600

Note: in this case each instalment is paid of Rs. 800 plus interest.

EXERCISE – 4.5

1- For each of the following.

(i) find the additional amount you have to pay by financing and

(ii) express the additional amount obtained in as a percentage of the cash price:

	Cash (Rs.)	Financing Term		
		Down (Rs.)	Monthly instalment (Rs.)	Number of instalments
(a)	Rs. 360	Rs. 50	Rs. 40	10
(b)	Rs. 900	Rs. 150	Rs. 75	12
(c)	Rs. 25000	Rs. 10000	Rs. 500	36

2- Pervaiz buys a window air-conditioner at *Rs.900*. He pays *20%* deposit and the outstanding balance plus markup in *48* months. Markup on the balance is charged at *10%*. Find

(i) the cost of his monthly instalment;

(ii) the amount he saves by paying cash.

3- On each of the following

(i) find the financial price of the goods and

(ii) express the amount saved by paying cash as a percentage of the cash price

	Item	Cash Rs.	Deposit	Number of Instalments	Monthly Instalments Rs.
(a)	Computer	Rs. 200	10%	24	Rs. 9
(b)	Printer	Rs. 450	15%	18	Rs. 25
(c)	Scanner	Rs. 1600	25%	30	Rs. 52

4- For each of the following, find

(i) the monthly instalment and

(ii) the difference in the hire purchase price and the cash price as a percentage of the cash price:

	Cash Rs.	Hire-purchase terms
(a)	Rs. 800	Rs.100 deposit; balance 8%; 1 year
(b)	Rs. 8000	Rs.200 deposit; balance 10%; $2\frac{1}{2}$ year
(c)	Rs. 1200	Rs.200 deposit; balance 15%; $1\frac{1}{3}$ year

5- The cash price of a computer package deal was *Rs.3200*. Markup paid @ *15%* down payment and the outstanding balance plus markup over *24* months. Markup on the balance was charged at *9.5%*.

(i) Find the cost of the package deal if it is bought on hire-purchase.

(ii) Find the difference between the hire-purchase price and the cash price.

(iii) Express the difference obtained in *(ii)* as a percentage of the cash price.

Review Exercise – 4

1- Encircle the correct answer.

(i) An instrument for payment order issued by a bank on the request of its customers is called:

(a) *pay order* (b) *cheque*
(c) *bank draft* (d) *bill of exchange*

(ii) The person or entity whose insurance is being done is called the:

(a) *insurer* (b) *insured*
(c) *drawer* (d) *lessee*

(iii) The company undertaking the act of insurance is called:

(a) *insurer* (b) *insured*
(c) *insurance* (d) *insurance policy*

(iv) The periodic instalment to be paid by the insured is called:

(a) *bonus* (b) *discount*
(c) *premium* (d) *mark up*

(v) The return earned by the bank on loan is named as:

(a) *mark up* (b) *premium*
(c) *bonus* (d) *profit*

(vi) The amount which is paid by the bank on the deposits is called:

(a) *profit* (b) *bonus*
(c) *premium* (d) *markup*

(vii) The percentage of profit/markup charged is called:

(a) *rate* (b) *time*
(c) *interest* (d) *principal*

(viii) A machine installed by the bank to dispense cash to customer is called an:

(a) *computer* (b) *scanner*

(c) *ATM* (d) *card reader*

(ix) A bill of exchange drawn on a specified banker and not expressed to be payable otherwise then on demand is called:

(a) *cheque* (b) *pay order*

(c) *bill of exchange* (d) *bank draft*

2- Fill in the blanks.

(i) A bill of exchange drawn on a specified banker and not expressed to be payable otherwise then on demand is called a ________

(ii) An instrument like a cheque, issued by bank on the request of its customers is called ________

(iii) A machine installed by the bank to dispense cash to customers is called an________

(iv) The amount which is paid by the bank on the deposits maintained by the client with the bank is called________

(v) The percentage of profit charged is called________

(vi) The period of the loan or deposit is called the________

(vii) The return earned by the bank on loan is named as ________

(viii) The periodic installment to be paid by the insured is called______

(ix) The company undertaking the act of insurance is called the______

(x) The person or entity whose insurance is being done is called the________

3- Raheel insured his house worth *Rs.75,00,000 @ 2%* for *5* years. Calculate the amount paid in *5* years, while the rate of depreciation is *10%* yearly.

4- Naeem insured his factory @ *3%* for *3* years. With depreciation rate *5%* yearly. If first premium is *Rs.21,000*, find the worth of the factory, if he got a claim of *Rs.200,000* after two years. How much benefit did he get?

5- M/s Rahim printer purchases under hire-purchase system a machine from Lahore company on *1st* January *2000*, paying cash *Rs.10,000* and agreeing to pay three further instalments of *Rs.10,000* each on *31st* December every year. The cash price of the machine is *Rs. 37,230* and the Lahore company charges markup at *5% p.a.* Draw table showing annual installment *(Principal + Markup)*.

SUMMARY

- A running account which continuously remains in operation due to its liquidity is called current account.
- Saving account is meant to encourage thrift and promote saving among the persons of small means. The bank pays nominal interest half yearly on the basis of monthly balance to the depositors.
- Profit and loss sharing (PLS) account is opened with small amount with profit earned or loss sustained at the end of each half year / full year depending upon the mode of payment.
- The deposits kept with the bank in an account for a certain period of time ranging term 3 months to 5 years is called fixed / time deposit account.
- Account maintained with the bank in foreign currency like dollars, pounds and Euro etc. is called foreign currency account.
- Negotiable instrument means a promissory note, a bill exchange or cheque payable whether to be ordered or bearer of the instruments.
- Insured is the person or entity whose insurance is being done is called "the insured".
- The company undertaking the act of insurance is called the insurer.
- A person to whom an insurance policy issued, the beneficiary in a contract issuance is called insured.
- The contract which is executed between two parties is called insurance policy.

- The periodic installment to be paid by the insured is called premium.
- The time-period agreed upon by both the parties (insured and insurer) is called maturity.
- The agreed amount to be paid back on maturity or expiry of the agreed period, includes the actual amount paid in installments with profit is termed as bonus.
- Cheque is a bill of exchange drawn on a specific banker and not expressed to be payable other wise on demand.
- Pay order is like a cheque issued by bank on the request of its customers.
- An order to pay money, drawn by one office of a bank upon another office of the same bank for a sum of money payable to order on demand is called bank draft.
- Online banking is the system where as a direct connection is made to centralized computer system for authorization or validation of transaction.
- ATM card is a payment card issued to a person for activating automated teller machine computer based terminal which allows consumers to make withdrawals.
- A card indicating that the holder has been granted a line of credit enabling the holder to make purchases and / or withdraw cash is called credit card.
- ATM machine installed by the bank to dispense cash to customer.

- Profit is the amount which is paid by the bank on the deposits maintained by the client with the bank.
- The amount / capital borrowed or lent is called principal.
- The percentage of interest charged is called rate.
- The period or duration of the loan or deposit is called the time.
- When the profit/markup is added to the principal, the sum is called the amount.
- The return earned by the bank is named as markup.
- Leasing is a contract where by the owner of an asset, the lessor, gives the hirer, the lessee, the right to use the asset for a specific period in exchange of rental payment.
- The customer is required to deposit the payment with the bank along with the application form is called down payment.

UNIT 5

CONSUMER MATHEMATICS

- **Taxes**
- **Utility Bills**
- **Personal Income**

After completion of this unit, the students will be able to:

- Know the term tax (direct and indirect).
- Explain the following in simplest possible terms:
  - Sales tax,
  - Excise duty,
  - Property tax,
  - Income tax,
- Calculate the amount of
  - Sales tax, levied on various commodities,
  - Excise duty, levied on different items,
  - Property tax, imposed on property,
  - Income tax, imposed on an individual with fixed income,
- Calculate amount of bill for:
  - Electricity,
  - Gas,
  - Telephone.
- When previous and present meter readings are given.
- Calculate personal income (weekly, monthly and annually) of
  - A worker who is paid on daily basis.
  - A worker who is paid for overtime on hourly basis in addition to his daily wages.
  - A salesman who is paid for overtime on hourly basis and commission on different sales in addition to his regular pay.
- Calculate gross income of a salaried person who is paid on the basis of government pay scales or otherwise.
- Calculate net income taking into account assorted deductions (income tax etc).

5.1 TAXES

To finance public spending on Education, Health, National Defence etc. Government imposes various taxes, which includes direct taxes and indirect taxes on its residents.

5.1.1 Tax

Money that must be paid to the State, charged as a proportion of income and profits or added to the cost of some goods and services is called a tax.

Direct Tax

This is the tax which is charged on income, property and profits in the form of income tax, property tax etc.

Indirect Tax

Indirect tax includes duties, motor vehicle taxes, goods and services taxes, general sales tax (GST) and value added taxes etc.

5.1.2 Key Terms

Sales Tax

When we buy an article, we have to pay a certain amount of tax as the value added tax in addition to the price of an article. This tax is usually given at a certain percentage of the selling price.

In Pakistan *16%* Sales tax is imposed on goods bought and services rendered.

EXAMPLE-1

A sales tax of 16% is imposed on television. If the marked price of the television is Rs. 18000. Calculate the total amount one has to pay, if he wants to purchase it.

SOLUTION: *Marked price of the television* $= Rs.18000$

$$\text{Sales tax payable} = \frac{16}{100} \times 18000 = Rs.2880$$

$$\text{The total amount one has to pay} = Rs.18000 + Rs.2880 = Rs.20880$$

EXAMPLE-2

A computer price is Rs. 34,800 inclusive of 16% sales tax. What is the original price of the computer ?

SOLUTION: Computer price including tax = Rs.34,800

Let the original price = Rs.100

Then the price of computer including sales tax @16% = Rs.116

$$\text{Therefore, original price} = \frac{100}{116} \times 34,800 = \frac{3480000}{116}$$

$$= Rs.\,30,000$$

Excise Duty

It is the form of a tax which the buyer pays on a manufactured item at the time of purchase.

For example while purchasing cars, Motor cycles, electronic appliances, cloth etc, one has to pay Excise duty along-with the selling price.

EXAMPLE-1

A man wants to purchase a car of 1000 cc. He has to pay 150 % excise duty on price of the car. If price of the car is Rs. 5,00,000. How much amount he has to pay to purchase the car ?

SOLUTION: Price of the car = Rs. 5,00,000 , Excise duty = 150%

$$\text{Amount of excise duty on Rs, 5,00,000} = \frac{150}{100} \times 5,00,000$$

$$= 150 \times 5000 = Rs.7,50,000$$

The man has to pay Rs. 7,50,000 + 5,00,000 = Rs. 12,50,000

EXAMPLE-2

The price of a television is Rs. 14040 which includes 20% excise duty. Find the amount of excise duty included in the price of the television. Also find the price of the television without duty.

SOLUTION: Price of the televison including Excise duty = Rs.14040

Excise duty = 20%

Suppose the original price of television = Rs.100

Price of television including 20% Excise duty = Rs.120

$$\text{Original price of television} = 14040\times\frac{100}{120}$$

$$= 14040\times\frac{5}{6}$$

$$= Rs.2340\times 5 = Rs.11700$$

Property Tax

A property tax is charged on the owner of land, houses, flats or buildings at a standard rate of *16%*, on the annual value of the property.

EXAMPLE

The annual income of a flat is Rs. 14,00,000. Find the tax payable at a rate of 16%.

SOLUTION: Annual income of flat = 14,00,000

Rate of tax = 16%

$$\text{Tax payable} = 14,00,000\times\frac{16}{100}$$

$$= 14,000\times 16$$

$$= Rs.2,24,000$$

Thus tax payable is Rs. 2,24,000

Income Tax

It is the tax charged on all incomes during the financial year from *1st* July to *30th* of June. If the income return is filed on *30th* June *2010*. Then the fiscal year considered is *2009-2010* and tax year is *2009*. At the end of each financial year one has to submit a return showing what was the earning during the year. The government exempts certain amount not to charge tax on it. The tax is charged on exceeding amount.

In order to work out the taxable income for salaried persons, all allowances are included in the basic pay.

EXAMPLE-1

The annual income of a person including all allowances is Rs. 3,60,000 and the exempted amount is Rs.1,80,000. Find the income tax payable by him at the rate of Rs.0.75%.

SOLUTION: *Total annual income* = Rs. 3,60,000

Exempted amount = Rs. 1,80,000

Taxable income = Rs. 3,60,000 − Rs. 1,80,000

= Rs. 1,80,000

Tax rate = 0.75 %

Thus tax payable = 1,80,000 × 0.75%

$$= 1,80,000 \times \frac{0.75}{100}$$

$$= 1,80,000 \times \frac{75}{100 \times 100}$$

$$= 18 \times 75$$

= Rs. 1350

EXAMPLE-2

The total annual income of a person is *Rs. 6,28,500* and the exempted amount is *Rs. 1,80,000*. Calculate the net income tax payable at the rate of *3.50%*. If the tax deducted at source is *Rs. 15,000*.

SOLUTION: *Total annual income* = Rs. 6,28,500

Exempted amount = Rs. 1,80,000

Taxable income = Rs. 6,28,500 − Rs. 1,80,000

= Rs. 4,48,500

Tax rate = 3.50 %

Total tax payable = Rs. 4,48,500 × Rs. 3.50%

$$= Rs.\,4,48,500 \times \frac{3.50}{100}$$

= Rs. 4485 × 3.50

= Rs. 15,697.50

Tax deducted at source = 15,000

Tax payable = Rs. 15,697.50 − Rs. 15,000

= Rs. 697.50

EXERCISE – 5.1

1- The price of a bicycle is *Rs. 3500*. If *16%* sales tax is charged, then calculate the amount of sales tax on *50* such bicycles.

2- If the price of an air conditioner is *Rs. 40,000*, then work out the amount of sales tax on it at the rate of *16%*. Also calculate the price of an air conditioner with sales tax.

3- The price of two cars of *1300 cc* and *1600 cc* without excise duty are *6,00,000* and *Rs. 8,00,000* respectively. If the excise duty on these two are *200%* and *250%* respectively. Find the prices of the two cars inclusive duties.

4- The annual price of a house and price of land is *Rs. 15,00,000* and *Rs. 20,00,000* respectively. Find the property tax on each of these two at the rate of *16%*.

5- The total taxable income of two persons is *Rs. 2,50,000* and *Rs. 3,10,000* respectively. Work out the income tax for each of them @ *4.5%*.

6- The total taxable income of a person is *Rs. 4,30,000*. If he is given rebate *Rs. 3000* on the tax chargeable, then work out the amount he has to pay as an income tax @ *4.5%*.

7- If the total annual income of a person is *Rs. 6,25,000* with exemption of amount of *Rs. 1,50,000*, then find the tax chargeable @ *4.5%*.

8- The total income of a person is *Rs. 5,25,000*. Whereas the exemption is *Rs. 1,50,000*. Work out the tax payable @ *4.5%* along with the income tax rate, if *Rs. 10,000* has already been deducted at source as income tax.

5.2 UTILITY BILLS

In this section we are to consider utility bills relating to electricity, gas and telephone only.

Electricity Bill

Domestic electricity bills are calculated by charging every house a fixed amount. This amount consists of variation of rates in units consumed. Rates of units are below:

First 100 units @Rs.2.65,

Second 200 units @Rs.3.64,

Next 700 units @Rs.6.15,

Remaining units @Rs.7.41.

The units consumed are recorded on a device called meter. The difference between the readings at the beginning and end of a month shows how much electricity has been used; Excise Duty, PTV Fee and Income Tax are also charged in electricity bill every month.

EXAMPLE

Ahsan uses 1050 units of electricity in a month. How much does electricity cost him for a month?

SOLUTION:

Number of units consumed	= 1050
Cost of 100 units @ Rs. 2.65 is:	$100 \times 2.65 = Rs. 265$
Cost of next 200 units @ Rs. 3.64 is:	$200 \times 3.64 = Rs. 728$
Cost of next 700 units @ Rs. 6.15 is:	$700 \times 6.15 = Rs. 4305$
Cost of remaining 50 units @ Rs. 7.41 is:	$50 \times 7.41 = Rs. 370.50$
Total cost of 1050 units is	Rs. 5668.50 .....(i)
Excise duty @ 1.5% is	Rs. 85.03 .....(ii)
Electricity duty	Rs. 62.52 .....(iii)
PTV fee	Rs. 25.00 .....(iv)
Income tax @ 1.6%	Rs. 91.00 .....(v)
Total cost (from (i) .... (v))	Rs. 5932.05 or Rs. 5932

Gas Bill

Many house hold use gas for cooking and heating. The amount of gas used is measured by volume and recorded by a meter in units, each of which *100* cubic feet of gas.
Gas slab rates for domestic purpose w.e.f *01-07-2009* are:

Where as $1\ Hm^3 \approx 3.30$ *MMBTU*

Slab	Usage of Gas in Hm^3	Rs. Per MMBTU
1	0 to 0.50	80.65
2	over 0.50 to 1	84.45
3	over 1 to 2	153.73
4	over 2 to 3	325.48
5	over 3 to 4	423.42
6	over 4 to 5	550.44
7	5 and above	730.17

EXAMPLE

The gas meter reading shows that $4.872\ Hm^3$ gas was used during a month. Work out the payable amount inclusive of GST @ *16%*.

SOLUTION: Gas charges $4\ Hm^3$ = Rs. 2201.76 ....(i)

$.872\ Hm^3$ = Rs. 479.98 ....(ii)

Meter rent = Rs. 120.00 ....(iii)

Total amount (i) + (ii) + (iii) = Rs. 2801.74

GST @ 16% $= \frac{16}{100} \times 2801.74$ = Rs. 448.27

Current bill = Rs. 2801.74 + Rs.448.27

= Rs. 3250.01

= Rs. 3250

Telephone Bills

The cost of telephone call depends upon three factors.

(i) The distance between the caller and the person being called.

(ii) The time of day and/or the day of the week on which the call is being made.

(iii) The length of the call.

These three methods are put together in various ways to give metered units of time, each unit being charged at a fixed rate.

The telephone bill of a consumer according to the new rules is as under:

(i) Net *PTCL* dues: ***(included line rent, local call units charges, NWD call units charges, local mobile unit charges, NWD mobile unit charges)***	*Rs. 1233*
(ii) Net *FED* *(Federal excise duty @ 21%)*:	*Rs. 259*
(iii) Net *W.H.Tax* *(withholding tax @ 4%)*:	*Rs. 49*
Total amount payable by due date:	*Rs. 1540*
Surcharge:	*Rs. 80*
Payable after due date:	*Rs. 1620.00*

The cellular phones bill is calculated as under:

Call charges : @ *Rs. 3* per minute: *(190 minutes)*	*Rs. 570.00*
SMS charges @ *Rs. 1* per SMS:	*Rs. 100.00*
CED *(Central excise duty @ 15%)*	*Rs. 105.00*
Total amount payable:	*Rs. 775.00*

EXERCISE – 5.2

1- In the following the gas meter reading has been given. Complete the gas bills with the help of the slabs given in the unit. Also include the meter rent and *GST*.

(i) $3.0756\ Hm^3$ (ii) $4.285\ Hm^3$ (iii) $2.796\ Hm^3$

(iv) $1.378\ Hm^3$ (v) $5.235\ Hm^3$ (vi) $4.665\ Hm^3$

2- In the following the number of units consumed while using electricity are given. Complete the Electricity bills, including the items as well as shown in the solved example of electricity bill.

(i) 315 units (ii) 210 units

(iii) 375 units (iv) 290 units

3- In the following the number of calls made are given. Complete the telephone bill including the items;
Call rate *Rs. 5* per call, FED @ *15%* W.H tax @ *4%*.

(i) 530 (ii) 640 (iii) 750

(iv) 270 (v) 480 (vi) 315

5.3 PERSONAL INCOME

The income earned by an individual while working on daily, weekly, monthly or annually basis is called personal income of a person.

5.3.1 Personal Income of a Worker (Who is Paid on Daily Basis)

To find the personal income of a person working on daily basis, we see the following examples.

EXAMPLE-1

Calculate the gross daily wage for each of the following factory workers, if the hourly rate of pay is Rs.50 and the day is consisting of 8 hours.

	Name	Number of hours worked	Pay at the rate of Rs.50 per hours
1	Aslam	3	$3\times 50 = Rs.150$
2	Anwer	5	$5\times 50 = Rs.250$
3	Daniyal	8	$8\times 50 = Rs.400$
4	Abdullah	7	$7\times 50 = Rs.350$
5	Ali	4	$4\times 50 = Rs.200$
6	Hamza	6	$6\times 50 = Rs.300$

Daily wages along with over time

EXAMPLE-2

Daniyal works five-day a week, Monday to Friday. He starts work every-day at 8:00am and finishes at 4:00pm. He has one hour off for lunch while he works 2 hours daily as overtime. If he is paid Rs.60 per hour as regular salary and Rs.80 per hour as over time, then how many hours does he work in a month? Find his gross pay per month as well.

SOLUTION:

Number of hours from 8:00am to 4:00pm = 8 hours.

As he has one hour off for a lunch,

therefore number of hours worked each day = 7.

$$\text{Number of hours woked in a week} = 5\times 7$$

$$= 35 \text{ hours}$$

Number of hours works in a month $= 4 \times 35$

$= 140 \text{ hours}$

Pay for a month $= 140 \times 60$

$= Rs. 8400$

He works 2 hours daily as over time, number of hours he works as over time in a week $= 2 \times 5$

$= 10 \text{ hours}$

Number of hours he works as over time in a month

$= 4 \times 10$

$= 40 \text{ hours}$

Over time payment for a month $= 40 \times 80$

$= Rs. 3200$

Gross pay $= Pay + Over\ Time\ payment$

$= Rs. 8400 + Rs. 3200$

Gross pay $= Rs. 11600$

EXAMPLE-3

A sales man on a shop is paid Rs. 60 per hour along with a 5% commission on the sale price of the articles he sells. If he works on the shop from 8am to 10pm and sells articles of worth Rs. 20,000, what is his daily gross income?

SOLUTION:

Number of hours from 8am to 10pm = 14.

Salary per hour = Rs. 60

Daily salary is $= Rs. 14 \times 60$

$= Rs. 840$

Commission $= 5\%$

$5\% \text{ of } 20{,}000 = \frac{5}{100} \times 20000$

$= 1000$

Net daily income $= Salary + Commission$

$= Rs. 840 + Rs. 1000$

$= Rs. 1840$

5.3.2 Gross Income of a Salaried Person

Gross income of a salaried person includes basic pay, house rent, conveyance allowance, dearness allowance, medical allowance etc,

EXAMPLE

If the basic salary of a person is Rs.30,000 per month and he is paid house rent at the rate of 15% of his basic salary and 10% as dearness allowance and Rs.2000 as conveyance allowance. Further he is paid Rs. 1000 as medical allowance, then calculate his gross monthly income.

SOLUTION:

$$\text{Basic salary} = Rs.\,30{,}000$$

$$\text{House rent} = 15\% \text{ of basic salary} = \frac{15}{100} \times 30000$$

$$= Rs.\,4500$$

$$\text{Dearness allowance} = 10\% \text{ of basic salary} = \frac{10}{100} \times 30000$$

$$= Rs.\,3000$$

Then monthly salary structure is:

$$\text{Basic pay} = Rs.\,30{,}000$$

$$\text{House rent} = Rs.\,4500$$

$$\text{Dearness allowance} = Rs.\,3000$$

$$\text{Conveyance allowance} = Rs.\,2000$$

$$\text{Medical allowance} = Rs.\,1000$$

$$\text{Gross monthly income} = Rs.\,40{,}500$$

5.3.3 Income after Deductions

The total income or take home salary of a person means the salary received after necessary deductions from the gross salary.

EXAMPLE

If the gross salary of a person is Rs. 25000, and he has to pay Rs. 360 per month as income tax, Rs. 1500 as benevolent fund and Rs. 1000 as G.P fund and Rs. 300 as group insurance, then work out his net (take home) income.

SOLUTION: Gross salary $= Rs. 25,000$

Deductions

$$\text{Income tax} = Rs. 360 ....(i)$$

$$\text{Benevolent fund} = Rs. 1500 ....(ii)$$

$$\text{G.P. fund} = Rs. 1000 ....(iii)$$

$$\text{Group insurance} = Rs. 300 ....(iv)$$

$$\text{Total deductions} = Rs. 3160 : \{sum\ of\ (i),\ (ii),\ (iii),\ (iv)\}$$

$$\text{Net income} = Gross\ salary - Total\ deductions$$

$$= Rs. 25000 - Rs. 3160$$

$$\text{Net income} = Rs. 21,840$$

EXERCISE – 5.3

1- A lady worker works a six-day week. She starts work at *7.00 am* and finishes at *4pm*. She has *15* minutes break in the morning and *45* minutes break in the afternoon. How long does she actually work in a week and how much she is paid, if the rate of payment is *Rs.40* per hour?

2- Khalid works *6* day-week. Find his gross monthly wage, if his rate of pay is *Rs. 200* per day.

3- Aslam gets paid *Rs. 70* per hour for his normal working 8 hours daily (*6* day week). The rate of over time is *1.5* of *Rs. 70* per hour. If he works *40* hours as overtime, then work out his gross monthly pay.

4- Calculate the gross monthly pay of a person, if his basic pay is *Rs.18000*, house rent allowances is *Rs,3500*, dearness allowances is *Rs.3000*, conveyance allowance is *Rs1500* and medical allowance is *Rs.500*.

5- If gross pay of a person is *Rs.45,000*, then calculate his net take home salary, after deductions of *Rs.400* as income tax, *Rs.1200* as benevolent fund, *Rs.1500* as G.P. fund and *Rs.400* as group insurance.

6- Noman works in a factory where the basic hourly rate is *Rs.50* for a *35* hour week. An over time is paid at time* and - a-half. How much will he earn in a week when he works for:

(i) *38 hours* (ii) *48 hours* (iii) *50 hours*

7- Abdullah's pay slip showed that he had worked *6* hours over time in addition to his basic *36* hours in a week. If his basic rate of pay is *Rs.60* and over time is paid at time and a-half. Find his gross pay for the month.

*Time and-a-half stands for = *1.5* hours.

Review Exercise – 5

1- Encircle the correct answer.

(i) Money that must be paid to the state charged as proportion of income and profit added to cost of some goods and services is called a

(a) *tax* (b) *excise duty* (c) *property tax* (d) *income tax*

(ii) The taxes which are charged on income, property and profits in the from of income tax, property tax and profits etc is called

(a) *tax* (b) *direct tax* (c) *property tax* (d) *income tax*

(iii) Taxes of the form of duties, motor vehicle taxes are called

(a) *indirect tax* (b) *direct tax* (c) *property tax* (d) *income tax*

(iv) The tax in addition to the price of the article is called

(a) *tax* (b) *sales tax* (c) *income tax* (d) *excise duty*

(v) The form of a tax which a buyer pays on a manufactured item at the time of purchase is called

(a) *excise duty* (b) *tax* (c) *income tax* (d) *sales tax*

(vi) The tax charged on the owner of a land, house flats or building is called

(a) *property tax* (b) *income tax* (c) *direct tax* (d) *indirect tax*

(vii) The tax charged on all the taxable income is called

(a) *sales tax* (b) *direct tax* (c) *income tax* (d) *excise duty*

2- Fill in the blanks.

(i) Money that must be paid to the state charged as a proportion of income and profits added to the cost of some goods and services is called a____________.

(ii) The taxes which are charged on income, property and profits in the form of income tax, property tax and profit etc is called a ____________.

(iii) Taxes of the form duties, motor vehicle taxes, goods and services are called ____________ .

(iv) The tax in addition to the price of the article is called as ______ .

(v) The form of a tax which a buyer pay on a manufactured item at the time of purchase is called ____________ .

(vi) The tax charged on the owner of a land, house, flats or building is called a ____________ .

(vii) The tax charged on all taxable income is called ____________ .

(viii) If the annual value of a flat is *Rs.6,00,000*. Then the tax payable at a rate of *15%* is ____________ .

(ix) The value added tax at the rate of *10%* at the marked price of television of *Rs.12000* is ____________ .

(x) The excise duty at rate of *150%*, one has to pay against an amount of *Rs.3,00,000* is ____________ .

3- The price of a tricycle is *Rs.4000*. If *16%* sales tax is charged, then calculate the amount of sales tax on *30* such tricycles.

4- If the total income of a person is *Rs.7,00,000* with exempted amount of *Rs.1,50,000*. Find the tax chargable @ *4.5%*.

5- The gas meter shows that $5.670\ Hm^3$ gas was used during a month period. Workout the payable amount inclusive of GST @ *16%*.

6- The number of units consumed while using electricity are as under.
(i) *275* units (ii) *200* units (iii) *340* units (iv) *285* units.
Complete the electricity bills, including the items as well as shown in the solved example of electricity bill.

7- The gross monthly pay of a person is *Rs.75,000*. If *Rs.1500*, *Rs.1200* and *Rs.1800* are deducted as income tax, benevolent find and G.P fund respectively, then calculate the net take home salary of the person.

SUMMARY

- Money that must be paid to the state, charged as a proportion of income and profits are added to the cost of some goods and services is called tax.

- Direct tax is charged on income, property and profits in the from of income tax, property tax and profit tax etc.

- Indirect taxes include duties, motor vehicle taxes, goods and services taxes, *(GST)* general sale tax and value added taxes etc.

- When we buy article we have to pay a certain amount of tax as the value added tax in addition to the price of the article is called sales tax. This tax usually given as a certain percentage of the selling price. In Pakistan sales tax of *16%* is imposed on goods bought and services rendered.

- Excise duty is the form of a tax which the buyer pay on a manufactured item at the time of purchase.

- A property tax is charged on the owner of land, house, flats or building at a standard rate of *16%* on the annual value of the property.

- Income tax charged on all taxable incomes during the year from *1st* July to next *30th* June.

UNIT

6

EXPONENTS AND LOGARITHMS

- Radicals and Radicands
- Laws of Exponents / Indices
- Scientific Natation
- Logarithm
- Laws of Logarithm
- Application of Logarithm

After completion of this unit, the students will be able to:

- Explain the concept of radicals and radicands.
- Differentiate between radical form and exponential form of an expression.
- Transform an expression given in radical form to an exponential form and vice versa.
- Recall base, exponent and value.
- Apply the laws of exponents to simplify expressions with real exponents.
- Express a number in standard form of scientific notation and vice versa.
- Define logarithm of a number to the base a as the power to which a must be raised to give the number ($a^x = y \Leftrightarrow \log_a y = x, a > 0, y > 0$ and $a \neq 1$)
- Define a common logarithm, characteristic and mantissa of log of a number.
- Use tables to find the log of a number.
- Give concept of antilog and use tables to find the antilog of a number.
- Prove the following laws of logarithm.
  - $\log_a (mn) = \log_a m + \log_a n$,
  - $\log_a \left(\frac{m}{n}\right) = \log_a m - \log_a n$,
  - $\log_a m^n = n \log_a m$.
- Apply laws of logarithm to convert lengthy processes of multiplication, division and exponentiation into easier processes of addition and subtraction, etc.

6.1.1 Radicals and Radicands

Let us consider a real number $\sqrt{5}$. We may write it as $5^{1/2}$. Here 5 is positive rational number, 2 is a positive integer and $\sqrt{5}$ is irrational, therefore $\sqrt{5}$ is a radical (or surd) of order 2. It is a quadratic surd as well.

Also consider a real number $\sqrt[3]{4}$ we may write it as $4^{1/3}$. Here 4 is a positive rational number, 3 is a positive integer and $\sqrt[3]{4}$ is irrational, therefore $\sqrt[3]{4}$ is a radical (or surd) of order 3. It is a cubical radical as well.

Therefore a radical (or surd) is an irrational number that contains an irrational square root $2\sqrt{3},\ 4+3\sqrt{5},\ 10-4\sqrt{6},\ \frac{\sqrt{2}}{5},\ \frac{9}{\sqrt{7}}$ are all radicals (or surds).

Let a be a real number and n be a positive integer, then a number which when raised to the power $\frac{1}{n}$, gives $a^{1/n}$, is called the nth root of a, written as $\sqrt[n]{a}$.

Thus $\sqrt{2}=2^{1/2}, \sqrt[3]{2}=2^{1/3}, \sqrt[4]{5}=5^{1/4}$ etc.

The symbol $\sqrt[n]{\ }$ is called the radical sign of index n. In $\sqrt[n]{a}$, a is called redicand.

$\sqrt{a}$ is called a radical of order 2,

$\sqrt[3]{a}$ is called a radical of order 3,

$\sqrt[4]{a}$ is called a radical of order 4 and

$\sqrt[n]{a}$ is called a radical of order n.

$(\sqrt{a}+\sqrt{b})$ and $(\sqrt{a}-\sqrt{b})$ are conjugate radicals (or surds) of order 2 to each other. The product of these two radicals is a rational number.

6.1.2 Radical Form and Exponential Form of an Expression

$\sqrt[3]{8}$ is the radical form of $(2^3)^{1/3}$ as the radical can be expressed with fractional exponents, therefore exponential form of $\sqrt[3]{8}$ is $(2^3)^{1/3}$ or 2.

The radical form of $5(3)^{1/2}$

$$= 5\sqrt{3}$$

From the above examples, we see that the laws of exponents are therefore applicable to radicals also. Thus for any positive integer 'n' and a positive rational number 'a' we have the following.

Radical form	Exponential form
(i) $(\sqrt[n]{a})^n = a$	$\left(a^{1/n}\right)^n = a$
(ii) $\sqrt[n]{ab} = \sqrt[n]{a}\,\sqrt[n]{b}$	$(ab)^{1/n} = a^{1/n}b^{1/n}$
(iii) $\sqrt[n]{\frac{a}{b}} = \frac{\sqrt[n]{a}}{\sqrt[n]{b}}$	$\left(\frac{a}{b}\right)^{1/n} = \frac{a^{1/n}}{b^{1/n}}$
(iv) $(\sqrt[n]{a})^m = \sqrt[n]{a^m}$	$(a^{1/n})^m = (a^m)^{1/n} = a^{m/n}$

EXAMPLE-1

Simplify: (i) $(a^3b^2)^{1/4} \times (a^{1/3}b)^{3/4}$ (ii) $x^{1/4} \div x^{2/3}$

SOLUTION:

(i) $(a^3b^2)^{1/4} \times (a^{1/3}b)^{3/4}$

$$= a^{3/4}b^{2/4} \times a^{3/12}b^{3/4}$$

$$= a^{3/4}b^{1/2} \times a^{1/4}b^{3/4}$$

$$= a^{3/4} \times a^{1/4} \times b^{1/2} \times b^{3/4}$$

$$= a^{3/4+1/4} \times b^{1/2+3/4}$$

$$= a^{4/4} \times b^{2+3/4}$$

$$= a^1 b^{5/4}$$

$$= a b^{5/4}$$

(ii) $x^{1/4} \div x^{2/3}$

$$= x^{1/4} \times \frac{1}{x^{2/3}}$$

$$= x^{1/4}.x^{-2/3}$$

$$= x^{1/4-2/3}$$

$$= x^{(3-8)/12}$$

$$= (x)^{-5/12}$$

$$= \frac{1}{x^{5/12}}$$

EXAMPLE-2

Simplify: (i) $\sqrt{(a^3b^2)^{1/4}(a^{1/3}b^{3/4})}$ (ii) $\sqrt{x^{1/4} \div x^{\frac{-2}{3}}}$

SOLUTION:

(i) $\sqrt{(a^3b^2)^{1/4}(a^{1/3}b^{3/4})}$

$$= \sqrt{a^{3/4} \times b^{2/4} \times a^{1/3} \times b^{3/4}}$$

$$= \sqrt{a^{9+4/12}\, b^{2+3/4}}$$

$$= \sqrt{a^{13/12}\, b^{5/4}}$$

(ii) $\sqrt{x^{1/4} \div x^{\frac{-2}{3}}}$

$$= \sqrt{\frac{x^{1/4}}{x^{-2/3}}}$$

$$= \sqrt{x^{1/4} \times x^{2/3}}$$

$$= \sqrt{x^{3+8/12}}$$

$$= \sqrt{x^{11/12}}$$

6.1.3 To Transform an Expression in Radical From to an Expression in Exponential Form and Vice Versa

Let us consider the following examples:

EXAMPLE-1

Express in exponential form:

(i) $\sqrt[4]{81a^{28}}$ (ii) $\sqrt[3]{27x^{18}}$ (iii) $\sqrt[3]{\frac{x^7y^9}{z^4}}$

SOLUTION: (i) $\sqrt[4]{81a^{28}}$

$$= (81\,a^{28})^{1/4}$$
$$= (3^4\,a^{28})^{1/4}$$
$$= 3^{4/4} \times a^{28/4}$$
$$= 3^1 \times a^7 \quad = 3\,a^7$$

(ii) $\sqrt[3]{27x^{18}}$

$$= (27\,x^{18})^{1/3}$$
$$= (3^3 \times a^{18})^{1/3}$$
$$= 3^{3/3} \times x^{18/3}$$
$$= 3^1 \times x^6$$
$$= 3\,x^6$$

(iii) $\sqrt[3]{\frac{x^7y^9}{z^4}}$

$$= \left(\frac{x^7\,y^9}{z^4}\right)^{1/3}$$
$$= \frac{x^{7/3}\,y^{9/3}}{z^{4/3}}$$
$$= \frac{x^{7/3}\,y^3}{z^{4/3}} \quad = x^{7/3}\,y^3\,z^{-4/3}$$

EXAMPLE-2

Simplify and give answer in radical form:

(i) $\sqrt{18}\times\sqrt[5]{64}$ (ii) $a^{1/2}\times a^{2/3}\div a^{3/4}$ (iii) $(a^{1/2}b^{2/3})^{3/4}\div(a^{2/5}b^{1/3})^{5/6}$

SOLUTION: (i) $\sqrt{18}\times\sqrt[5]{64}$

$$= (18)^{1/2}\times(64)^{1/5}$$
$$= (9\times 2)^{1/2}\times(2^6)^{1/5}$$
$$= 9^{1/2}\times 2^{1/2}\times 2^{6/5}$$
$$= 9^{1/2}\times 2^{17/10}$$
$$= (9)^{1/2}\times(2^{17/5})^{1/2}$$
$$= \sqrt{9\times 2^{17/5}}$$

(ii) $a^{1/2}\times a^{2/3}\div a^{3/4}$

$$= a^{1/2+2/3-3/4}$$
$$= a^{6+8-9/12}$$
$$= a^{14-9/12}$$
$$= a^{5/12} = \sqrt[12]{a^5}$$

(iii) $(a^{1/2}b^{2/3})^{3/4}\div(a^{2/5}b^{1/3})^{5/6}$

$$= a^{1/2\times 3/4}\times b^{2/3\times 3/4}\div a^{2/5\times 5/6}\,b^{1/3\times 5/6}$$
$$= (a^{3/8}b^{1/2})\div a^{1/3}\,b^{5/18}$$
$$= a^{3/8-1/3}\,b^{1/2-5/18} = a^{9-8/24}\,b^{9-5/18}$$
$$= a^{9-8/24}\,b^{9-5/18} = a^{1/24}\,b^{4/18}$$
$$= \sqrt[24]{a}\ \sqrt[9]{b^2}$$

EXERCISE – 6.1

1- Determine the radicals and the radicands from the following.

(i) $\sqrt{3}$ (ii) $4+3\sqrt{a}$ (iii) $\sqrt{11}$

(iv) $8-2\sqrt{6}$ (v) $\frac{\sqrt{5}}{7}$ (vi) $\frac{9}{\sqrt{13}}$

2- Express the following in exponential form:

(i) $\sqrt{a^3}$ (ii) $\sqrt[5]{a^3}$ (iii) $\frac{1}{\sqrt[p]{a^k}}$ (iv) $\frac{1}{\sqrt[b]{a^k}}$

3- Write in the radical form and evaluate the result.

(i) $(25)^{1/2}$ (ii) $(64)^{1/3}$ (iii) $(81)^{1/4}$ (iv) $(27)^{1/3}$

(v) $(27)^{2/3}$ (vi) $8^{-1/3}$ (vii) $(1000)^{2/3}$ (viii) $(64)^{1/2}$

4- Simplify and give answer in exponential form.

(i) $\sqrt{a^{16}}$ (ii) $\sqrt[3]{a^{15}}$ (iii) $\sqrt[3]{27a^9}$ (iv) $\sqrt[3]{8a^9}$

(v) $\sqrt[4]{x^{32}}$ (vi) $\sqrt[4]{81x^{20}}$ (vii) $\sqrt[3]{125x^9y^{15}}$ (viii) $\sqrt{(8+y)^7}$

(ix) $\sqrt[4]{16x^2y^6}$ (x) $\sqrt[4]{\frac{x^5y^6}{z^2}}$ (xi) $\sqrt[3]{\frac{8x}{x+y}}$ (xii) $\sqrt[p]{\frac{y^n}{a^m}}$

5- Simplify:

(i) $\sqrt{3}\times\sqrt{7}$ (ii) $\sqrt[5]{4}\times\sqrt[5]{128}$ (iii) $\sqrt[5]{81}\times\sqrt[5]{27}$

(iv) $\sqrt{2}\div\sqrt[9]{32}$ (v) $\sqrt[5]{118}\div\sqrt[5]{2}$ (vi) $\sqrt{27}\div\sqrt{81}$

(vii) $a^{1/4}\times a^{2/3}$ (viii) $x^{6/7}\times y^{1/4}$ (ix) $(x^{3/4}y^{1/6})^6$

(x) $(x^3y^2)^{1/2}\times(y^3x^4)^{-1/3}$ (xi) $(x^3y^2)^{1/4}\times(x^{1/3}y)^{3/4}$

(xii) $(a^{1/4}b^{1/3})^{-1/2}\div(a^{1/3}b^{1/4})^{-5}$ (xiii) $(x^2y^3)^{1/5}\times(x^{1/3}y^2)^{1/4}$

6.2 LAWS OF EXPONENTS/INDICES

6.2.1 Base, Exponent and Value

Sometimes we experience a multiplication of the type:

$$3\times 3,\ 3\times 3\times 3,\ 3\times 3\times 3\times 3,\ 3\times 3\times 3\times 3\times 3$$

In simplified from we can write:

$$3\times 3 = 3^2$$
$$3\times 3\times 3 = 3^3$$
$$3\times 3\times 3\times 3 = 3^4$$
$$3\times 3\times 3\times 3\times 3 = 3^5 \quad \text{and so on.}$$

For any real number 'a' and a positive integer 'n' we define:

$$a^n = a\times a\times a\times a \cdots \times a \ (n \text{ times})$$

Here 'a' is called the base and 'n' is called the exponent or index.
By definition, we take $a^o = 1$, thus $2^o = 1, 3^o = 1, (0.5)^o = 1$ and so on.

Note: "a^n" is called as the n^th^ power of 'a'.

e.g

$$3\times 3\times 3 = 3^3$$
$$4\times 4\times 4\times 4\times 4 = 4^5$$
$$7\times 7\times 7\times 7 = 7^4$$
$$8\times 8 = 8^2$$

6.2.2 Laws of Exponents and their Applications

1- Law of Sum of Powers

If $a \in R,\ a \neq 0$ and $m, n \in \mathbb{Z}$, then

$$a^m \times a^n = a^{m+n}$$

EXAMPLE-1

Simplify: $x^3 \times x^4 \times x^6$

SOLUTION: $x^3 \times x^4 \times x^6 = x^{3+4+6}$

$= x^{13}$

EXAMPLE-2

Simplify: $x^3 \times y^4 \times x^4 \times y^3 \times x^5 \times y^5$

SOLUTION: $x^3 \times y^4 \times x^4 \times y^3 \times x^5 \times y^5 = x^3 \times x^4 \times x^5 \times y^3 \times y^4 \times y^5$

$= x^{3+4+5} \times y^{3+4+5}$

$= x^{12} \times y^{12}$

$= x^{12}\, y^{12}$

EXAMPLE-3

Simplify: $x^3 \times y^4 \times x^{-2} \times y^{-2}$

SOLUTION: $x^3 \times y^4 \times x^{-2} \times y^{-2} = x^3 \times x^{-2} \times y^4 \times y^{-2}$

$= x^{3-2} \times y^{4-2}$

$= xy^2$

2- Laws of Subtraction of Powers

If $a \in R, a \neq 0$ and $m, n \in \mathbb{Z}$, then

$$\frac{a^m}{a^n} = a^{m-n}$$

There are three cases:

Case I

When $m > n$.

$$\frac{a^m}{a^n} = \frac{a \times a \times a \times \cdots \text{ to } m \text{ factors}}{a \times a \times a \times \cdots \text{ to } n \text{ factors}}$$

$= a \times a \times a \times \cdots$ to $(m-n)$ factors

$= a^{m-n}$

Case II

When $m = n$.

In this case $\frac{a^m}{a^n} = \frac{a^m}{a^m} = \frac{a \times a \times a \times \cdots \text{to } m \text{ factors}}{a \times a \times a \times \cdots \text{to } m \text{ factors}}$

$$= 1$$

$$= a^o$$

$$= a^{m-m}$$

$$= a^{m-n} \; [\because m = n]$$

Definition of 'a^{-n}'

> We define $a^{-n} = \frac{1}{a^n}$, when $n \in \mathbb{Z}$ and $a \in R, a \neq 0$.

Case III

When $m < n$.

in this case $\frac{a^m}{a^n} = \frac{a \times a \times a \times \cdots \text{to } m \text{ factors}}{a \times a \times a \times \cdots \text{to } n \text{ factors}}$

$$= \frac{1}{a \times a \times a \times \cdots \text{to } (n-m) \text{ factors}}$$

$$= \frac{1}{a^{n-m}}$$

$$= a^{-(n-m)}$$

$$= a^{-n+m}$$

$$= a^{m-n}$$

Hence $\frac{a^m}{a^n} = a^{m-n}$, when $m > n$ or $m = n$ or $m < n$.

EXAMPLE-1

Simplify: (i) $\dfrac{x^3 \times x^5 \times x^6}{x^2 \times x^4 \times x}$ (ii) $\dfrac{x^3 \times x^4}{x^2 \times x^5}$ (iii) $\dfrac{x^4 \times x^5 \times x^6}{x^5 \times x^6 \times x^8}$

SOLUTION:

(i) $\dfrac{x^3 \times x^5 \times x^6}{x^2 \times x^4 \times x} = \dfrac{x^{3+5+6}}{x^{2+4+1}}$

$= \dfrac{x^{14}}{x^7}$

$= x^{14-7}$

$= x^7$

(ii) $\dfrac{x^3 \times x^4}{x^2 \times x^5} = \dfrac{x^{3+4}}{x^{2+5}}$

$= \dfrac{x^7}{x^7}$

$= x^{7-7}$

$= x^0$

$= 1$

(iii) $\dfrac{x^4 \times x^5 \times x^6}{x^5 \times x^6 \times x^8} = \dfrac{x^{4+5+6}}{x^{5+6+8}}$

$= \dfrac{x^{15}}{x^{19}}$

$= x^{15-19}$

$= x^{-4}$

EXAMPLE-2

Simplify: (i) $x^{1/5} \times x^{2/5}$ (ii) $(x^2 y^3)^{1/6}$ (iii) $\left(\dfrac{x^{2/3}}{y^{3/4}}\right)^{1/2}$

SOLUTION:

(i) $x^{1/5} \times x^{2/5}$

$= x^{1/5+2/5}$

$= x^{\frac{1+2}{5}}$

$= x^{\frac{3}{5}}$

(ii) $(x^2 y^3)^{1/6}$

$= x^{2/6} \times y^{3/6}$

$= x^{1/3} y^{1/2}$

(iii) $\left(\dfrac{x^{2/3}}{y^{3/4}}\right)^{1/2}$

$= \dfrac{(x^{2/3})^{1/2}}{(y^{3/4})^{1/2}}$

$= \dfrac{x^{1/3}}{y^{3/8}}$

3- Law of Power of Product

If $a, b \in R$, $a \neq 0$ and $n \in Z$, then:

$$(i)\ (ab)^n = a^n b^n$$

$$(ii)\ \left(\frac{a}{b}\right)^n = \frac{a^n}{b^n}$$

EXAMPLE

Simplify: (i) $(xy)^3$ (ii) $(xy)^6$ (iii) $\left(\frac{x}{y}\right)^5$ (iv) $\left(\frac{x}{y}\right)^4$

SOLUTION: (i) $(xy)^3 = x^3y^3$

(ii) $(xy)^6 = x^6y^6$

(iii) $\left(\frac{x}{y}\right)^5 = \frac{x^5}{y^5}$

(iv) $\left(\frac{x}{y}\right)^4 = \frac{x^4}{y^4}$

4- Law of Power of Power

If $a \in R, a \neq 0$ and $m, n \in Z$, then:

$$(a^m)^n = a^{mn}$$

EXAMPLE

Simplify: (i) $(x^3)^4$ (ii) $(x^4)^6$

SOLUTION:

(i) $(x^3)^4$
$= x^{3\times4}$
$= x^{12}$

(ii) $(x^4)^6$
$= x^{4\times6}$
$= x^{24}$

EXERCISE – 6.2

1- Write the base and exponent in the following.

(i) $16x^3$ (ii) x^9 (iii) $(4y)^3$

(iv) $(x-2)^3$ (v) $18x^5$ (vi) $5x^{3/2} \times x^{1/2}$

Simplify and express with positive indices

2- $\sqrt{(a^2b^3)^6}$ 3- $\sqrt[9]{(x^{-4}y^3)^{-3}}$ 4- $(x^a y^{-b})^3 \times (x^3y^2)^{-a}$

5- $\left(\frac{16x^2}{y^{-2}}\right)^{-1/4}$ 6- $\left(\frac{27x^3}{8a^{-3}}\right)^{-2/3}$ 7- $\left(\frac{a^{-1/2}}{4c^2}\right)^{-2}$

8- $\sqrt{a^{-2}b} \times 3\sqrt{ab^{-3}}$ 9- $\left(\frac{a^{-3}}{b^{-2/3}c}\right)^{-3/2} \div \frac{ab^2c}{a^2c}$ 10- $\frac{(a^4)^3(a^{-1}b)^{10}}{a^2b^7}$

11- $\frac{(x^3y)^3(2xy)^{-2}}{4x^{-4}y^{-5}}$ 12- $\frac{(a^{-5})^3 \times (ab)^{15}}{a^{-1}b^2}$ 13- $a^5b^4c^2 \div abc$

14- $(2ab^2)^2(3abc^2)^{-2} \div (ab)^{-4}(bca)^5$ 15- $\frac{2^3 \times 6^5}{3^{-3} \times 4^{-4}}$

16- $\frac{2^5 \times 9^{-1}}{27^{-3} \times 8^{-3}}$ 17- $(2^{-3}a^4b)^{-1} \times (4^{-2}b^{-5})$

Evaluate

18- $(3^2)^5 \div (9^3 \times 27^{-1})$ 19- $\left(\frac{3}{4}\right)^{-2} \div \left(\left(\frac{4}{9}\right)^3 \times \left(\frac{27}{16}\right)^{-1}\right)$

20- $\left(\frac{2}{3}\right)^{-1} \div \left(\left(\frac{4}{9}\right)^{-2} \times 27\right)$ 21- $\frac{5^4}{3^7} \times \left(\frac{9}{15}\right)^3 \div \frac{27}{25}$

22- $a^{1/2}b^{2/3} \times a^{2/3}b^{1/4}$ 23- $a^{2/3}b^{5/6} \times a^{1/2}b \div (ab)^{1/3}$

24- $(a^{1/2}b^{1/3}c^{1/4})^6$ 25- $(a^{1/2}b^{1/3})^{4/3} \div (a^{1/3}b^{1/4})^{1/2}$

26- $a^{2/3} \times a^{1/2} \div a^{1/4}$

27 Simplify each of the following.

(i) $4^{3/5} \times 4^{1/5}$ (ii) $2^{1/8} \times 2^{3/8}$

(iii) $5x^{1/3} \times 2x^{1/5}$ (iv) $x^{3/4} \times x^{2/5}$

(v) $\frac{1}{2}y^{3/7} \times 4y^{2/7}$ (vi) $5x^{3/2} \times x^{1/2}$

28 Simplify each of the following.

(i) $a^{2/3}b^{3/4} \times a^{1/3}b^{3/4}$ (ii) $x^{3/5}y^{2/9} \times x^{1/5}y^{1/3}$

(iii) $2ab^{1/3} \times 3a^{3/5}b^{4/5}$ (iv) $6x^{3/7} \times \frac{1}{3}x^{1/4}y^{2/5}$

(v) $x^3y^{1/2}z^{1/3} \times x^{1/6}y^{1/3}z^{1/2}$

29 Simplify each of the following.

(i) $3^{1/2} \div 3^{1/3}$ (ii) $\frac{x^{4/5}}{x^{5/9}}$ (iii) $\frac{2x^{3/4}}{4x^{3/5}}$

(iv) $\frac{25y^{3/5}}{20y^{1/4}}$ (v) $x^3y^2 \div x^{4/3}y^{3/5}$ (vi) $a^{5/9}b^{2/3} \div a^{2/5}b^{2/5}$

(vii) $10x^{4/5}y \div 5x^{2/3}y^{1/4}$ (viii) $\frac{5a^{3/4}b^{3/5}}{20a^{1/5}b^{1/4}}$

6.3 SCIENTIFIC NOTATION

In some branches of science we use very large and very small numbers. The speed of light is *186000* miles *(or 299337.24 km)* per second or *30,000,000,000* centimeters per second and the radius of a Hydrogen atom, i.e *0.000000073 cm* are the examples of very large and very small numbers respectively. The wave length of an *X-ray*, is *0.0000001* centimeter is also an example of a very small number.

An easy method is devised to write these numbers is known as "scientific notation".

In this method a number *'a'* can be written as the product of two numbers in which the first number is in between *0* and *10* and the second number is the positive or negative exponent of *10* i.e.

$$a = b \times 10^n$$

EXAMPLE-1

Write the following in scientific notation.

(i) 100 (ii) 1000 (iii) 10000 (iv) $\frac{1}{1000}$ (v) $\frac{1}{10000}$

SOLUTION:
(i) $100 = 1 \times 10^2$
(ii) $1000 = 1 \times 10^3$
(iii) $10000 = 1 \times 10^4$
(iv) $\frac{1}{1000} = 1 \times 10^{-3}$
(v) $\frac{1}{10000} = 1 \times 10^{-4}$

EXAMPLE-2

Write the following in scientific notation.

(i) 90.85 (ii) 112.3 (iii) 12.35 (iv) 0.00018 (v) 0.0000281

SOLUTION: (i)

$$\begin{aligned} 90.85 &= \frac{9085}{100} \\ &= 9085 \times 10^{-2} \\ &= 9.085 \times 10^3 \times 10^{-2} \\ &= 9.085 \times 10^{3-2} \\ &= 9.085 \times 10^1 \end{aligned}$$

(ii) 112.3

$$= \frac{1123}{10}$$
$$= 1123 \times 10^{-1}$$
$$= 1.123 \times 10^{3} \times 10^{-1}$$
$$= 1.123 \times 10^{2}$$
$$= 1.123 \times 10^{2}$$

(iii) 12.35

$$= \frac{1235}{100}$$
$$= 1235 \times 10^{-2}$$
$$= 1.235 \times 10^{+3} \times 10^{-2}$$
$$= 1.235 \times 10^{+3-2}$$
$$= 1.235 \times 10$$

(iv) 0.00018

$$= \frac{18}{100000}$$
$$= 18 \times 10^{-5}$$
$$= 1.8 \times 10 \times 10^{-5}$$
$$= 1.8 \times 10^{-4}$$

(v) 0.0000281

$$= \frac{281}{10000000}$$
$$= 281 \times 10^{-7}$$
$$= 2.81 \times 10^{2} \times 10^{-7}$$
$$= 2.81 \times 10^{2-7}$$
$$= 2.81 \times 10^{-5}$$

In scientific notation a positive number is written as the product of two numbers. In this, first number is obtained by placing decimal after the first digit of the given number.

For the second number to get the exponent of *10* we count the number of digits which is between the actual decimal place and the new place. If the decimal place is changed from the left side then the exponent of *10* is positive, while changing from the right side the exponent of *10* is negative.

EXAMPLE-1

Write 18.42×10^{-4} in decimal form.

SOLUTION: 18.42×10^{-4}

$$=\frac{1842}{100}\times10^{-4}$$

$$=\frac{1842}{100\times10^{4}}$$

$$=\frac{1842}{1000000}$$

$$=0.001842$$

EXAMPLE-2

Write 50,000,000 in scientific notation.

SOLUTION: $50,000,000$

$$=5\times10000000$$

$$=5\times10^{7}$$

EXERCISE – 6.3

Write the following in scientific notation:

1- 0.051
2- 89.99
3- 0.424
4- 2566324
5- 0.00000075

Write the following in the decimal form:

6- 0.86×10^{4}
7- 1.345×10^{-5}
8- 5.1×10^{-9}
9- 0.525×10^{-7}
10- 636.5×10^{-6}

Simplify and write your answer in scientific notation:

11- $\dfrac{0.96\times10^{7}}{2\times10^{4}}$
12- $\dfrac{2.61\times4\times10^{8}}{10^{3}}$
13- $\dfrac{521\times10^{3}\times12}{2\times10^{2}}$

14- Convert $4.5\times10^{5}\,cm$ into meters and write the solution in decimal form.

15- The radius of earth is $6400\ km$. Convert it into meters and write the solution in scientific notation.

6.4 LOGARITHM

Al-Khawarzmi contributed a lot towards logarithm. In *17th* century John Napier made further amendments in logarithm and prepared tables for it. He fixed a base *'e'* for these tables. The value of *'e'* is *2.7183*. John Napier and Henry Briggs made a plan to prepare table having base *10*. Later on, Henry Briggs completed the task and prepared tables with base *10*.

Jobst Burgi from Switzerland in *1620 A.D* prepared a table for anti-logarithm. These tables made the complicated problems easier regarding the counting of numbers.

6.4.1 Logarithm of a Number

Let $a > 0$ and $a \neq 1$, if *'y'* is any positive number, then:

$$x = \log_a y, \text{ if and only if } a^x = y$$

$$\text{or} \quad a^x = y \Leftrightarrow \log_a y = x \quad \ldots\ldots\ldots\ldots\ldots (1)$$

(For $\log_a y$, read "the logarithm of y to the base a".)

EXAMPLE-1

Convert the following into exponential form:

(i) $\log_5 25 = 2$ (ii) $\log_3 \frac{1}{9} = -2$ (iii) $\log_{10} 1000 = 3$

SOLUTION: Using the equation $\log_a y = x \Leftrightarrow a^x = y$, we have

(i) $\log_5 25 = 2$ is $5^2 = 25$

(ii) $\log_3 \frac{1}{9} = -2$ is $3^{-2} = \frac{1}{9}$

(iii) $\log_{10} 1000 = 3$ is $10^3 = 1000$

EXAMPLE-2

Solve the equation $\log_3(x+1) = 2$

SOLUTION: Using $\log_a y = x \Leftrightarrow a^x = y$, we have

$$3^2 = x + 1 \quad or \quad x + 1 = 9$$

$$\Rightarrow \quad x = 8$$

6.4.2 Common Logarithm

The logarithm calculated to the base 10 are called common logarithms. We denote $log_{10}m$ by log m only.

Clearly $10^1 = 10 \Leftrightarrow \log 10 = 1$; $10^2 = 100 \Leftrightarrow \log 100 = 2$

$$10^3 = 1000 \Leftrightarrow \log 1000 = 3. etc.$$

$$10^{-1} = \frac{1}{10} = 0.01 \Leftrightarrow \log(0.1) = -1,$$

$$10^{-2} = \frac{1}{100} = 0.1 \Leftrightarrow \log(0.01) = -2$$ and so on.

EXAMPLE

Solve (i) $\log\ (x-2) = 1$ (ii) $\log\ (x+3) = 2$

SOLUTION:

Using $log_a y = x \Leftrightarrow a^x = y$, we have

(i) $\log\ (x-2) = 1 \Rightarrow 10^1 = x-2 \Rightarrow x-2 = 10$

$\Rightarrow \boxed{x = 12}$

(ii) $\log\ (x+3) = 2 \Rightarrow x+3 = 10^2 \Rightarrow x+3 = 100$

$\Rightarrow \boxed{x = 97}$

Characteristic and Mantissa of a Log of a Number

The logarithm of a number consists of two parts, the integral part is known as the characteristic and the decimal part is known as the mantissa.

The mantissa is always taken as positive while the characteristic may be zero positive or negative. When the characteristic is negative, we put a bar on the digit representing characteristic, that instead of -2 we write it as $\bar{2}$; $\bar{2}.7638$ means $-2+0.7638$.

6.4.3 Finding the Logarithm of a Number

Characteristic of a Number

Write the number in standard form. Let it be $m \times 10^{p}$, then the characteristic is p, or

(i) The characteristic of a number greater than or equal to 1 is *one* less than the number of digits to the left of the decimal point in the given number.

(ii) The characteristic of a number less than 1 is a negative number whose numerical value is one more than the number of zeros between the decimal and the first significant digit of the number.

For example:

Number	Standard Form	Characteristic
5376.4	5.3764×10^{3}	3
537.64	5.3764×10^{2}	2
53.764	5.3764×10^{1}	1
5.3764	5.3764×10^{0}	0
0.5376	5.376×10^{-1}	$\bar{1}$
0.0537	5.37×10^{-2}	$\bar{2}$
0.00537	5.37×10^{-3}	$\bar{3}$
0.0000046	4.6×10^{-6}	$\bar{6}$

Mantissa of a Number

We find the mantissa from the log-table. The position of a decimal point in a number is immaterial for finding the mantissa. We restrict ourselves to the mantissa of a number consisting of four digits only.

e.g. $log(45)$, $log(.45)$, $log(.045)$, $log(.0045)$ etc have the same mantissa.

(i) For finding the mantissa of *4385* from log table, we proceed in the row headed by *43* and in this row, we find the number under the column headed by *8*. Now to this number we add the mean difference headed by *5*, in the same row.

Thus mantissa for *4385* is $.(6415 + 5) = .6420$.

(ii) For finding the mantissa of *438* from log-talbe, we find the number in the row headed by *43* and under the column by *8*. It is *.6415*.

(iii) For finding the mantissa of *43* from log-table, we find the number in the row headed by *43* and under the column *0*. It is *.6335*.

(iv) For finding the mantissa of *4* from log-table, we find the number in the row headed by *40* and under the column *0*. It is *.6021*.

Thus we have,

$\log 4385 = 3.6420$ $\log 0.4385 = \bar{1}.6420$

$\log 438.5 = 2.6420$ $\log 0.04385 = \bar{2}.6420$

$\log 43.85 = 1.6420$ $\log 0.004385 = \bar{3}.6420$

$\log 4.385 = 0.6420$ $\log 0.0004385 = \bar{4}.6420$

and $\log 43 = 1.6415$

$\log 4.3 = 0.6415$

Also $\log 4 = 0.6021$

$\log .04 = \bar{2}.6021$

6.4.4 Concept of Antilogarithm

If $\log m = n$, then m = antilog n, e.g $\log 1000 = 3 \Rightarrow antilog\ 3 = 1000$

For finding antilogarithm of a number, we use the decimal part of the number and read the antilog-table in a manner similar to that adopted for reading the log-table.

After finding the corresponding number from the antilog-table insert the decimal point as under:

Case I

When the characteristic is *'n'*, the decimal point is inserted after *(n+1)th* digit.

(i) Number $= 0.2346$

Characteristic $= n = 0$

Mantissa $= .2346$

From antilog table, the number against the mantissa *.2346* is *1724*. As the characteristic is *'0'* i.e. $n = 0$, therefore decimal point is inserted after $(0 + 1)th$ or first digit from the left of the number *1724*.

Thus antilog of $0.2346 = 1.724$

(ii) Number $= 2.6019$

Characteristic $= n = 2$

Mantissa $= .6019$

From antilog table, the number against the mantissa *0.6019* is *39908*. Decimal point is inserted after $(2 + 1)th$ digit or *3rd* digit from the left of the number.

Thus antilog of $2.6019 = 399.08$

(iii) Number $= 5.2612$

Characteristic $= n = 5$

Mantissa $= .2612$

From antilog table, the number against the mantissa *0.2612* is *1825*. Decimal point is inserted after $(5 + 1)th$ digit or *6th* digit from the left of the number *1825*.

Thus antilog of $5.2612 = 182500$

There are *4* digits in *1825*, therefore we put two zeros to the right of the number to make it a six digit number. Placing decimal after these zeros is meaningless.

Case II

When the characteristic is $\bar{n}$, the decimal point is inserted in such a way that the first significant figure is at the *nth* place.

(i) Number $= \bar{1}.4356$

Characteristic $= \bar{n} = \bar{1}$

Mantissa $= .4356$

From antilog table, the number against the mantissa *0.4356* is *2727*.

Since $\bar{n} = \bar{1}$, therefore the first significant figure will be at the *1st* place after decimal.

Thus antilog of $\bar{1}.4356 = 0.2727$

(ii) Number $= \bar{3}.1459$

Characteristic $= \bar{n} = \bar{3}$

Mantissa $= .1459$

From antilog table, the number against the mantissa *0.1459* is *1399*.

Since $\bar{n} = \bar{3}$, therefore the first significant figure will be at the *3rd* place after decimal.

Thus antilog of $\bar{3}.1459 = 0.001399$

EXAMPLE-1

Find the values of: (i) anti log 0.654 (ii) anti log 1.204

(iii) anti log $\bar{1}.3612$ (iv) anti log $\bar{3}.4568$

SOLUTION: (i) anti log (0.654) = 4.508

(ii) anti log (1.204) = 16.00

(iii) anti log ($\bar{1}.3612$) = 0.2297

(iv) anti log ($\bar{3}.4568$) = 0.002863

EXAMPLE-2

(i) Add $\bar{1}.3612$, 3.1946, $\bar{2}.0018$ and $\bar{3}.4619$

(ii) Subtract $\bar{4}.6342$ from 2.1375

(iii) Multiply $\bar{3}.4103$ with 6

(iv) Divide $\bar{5}.1820$ by 15

SOLUTION: (i) $\bar{1}.3612+3.1946+\bar{2}.0018+\bar{3}.4619$

$= -1+0.3612+3.1946-2+0.0018-3.4619$

$= -6+(.3612+3.1946+.0018+.4619)$

$= -6+(4.0195)$

$= -2+0.0195$

$= \bar{2}.0195$

(ii) $2.1375-\bar{4}.6342 = 2.1375-[-4+.6342]$

$= 2.1375+4-.6342$

$= 6.1375-.6342$

$= 5.5033$

(iii) $\bar{3}.4103\times 6 = (-3+.4103)\times 6$

$= -18+2.4618$

$= (-18+2)+.4618$

$= -16+.4618$

$= \overline{16}.4618$

(iv) $(\bar{5}.1820)\div 15 = (-5.1820)\div 15$

$= (-5+.1820)\div 15$

$= -4.9180\div 15$

$= -0.3212$

$= -1+(1-(.3212))$

$= \bar{1}.6788$

EXAMPLE-3

(i) If $\log x = 0.5019$, find x

(ii) If $\log x = \bar{2}.5321$, find x

SOLUTION: (i) $\log x = 0.5019$

Characteristic of $\log x = 0$

Mantissa of $\log x = 0.5019$

Now from the antilogarithm table, we have the number for mantissa
$0.5019 = 3170 + 7 = 3177$

As the characteristic is zero, the decimal point is inserted after $(0+1)th$ digit, i.e. after 3.

$$x = antilog\ (0.0519)$$
$$x = 3.177$$

(ii) $\log x = \bar{2}.5321$

Characteristic of $\log x = \bar{2}$

Mantissa of $\log x = .5321$

Now from the antilogarithm table, we have the number for mantissa
$.5321 = 3404 + 1 = 3405$

As the characteristic is $\bar{2}$, the decimal point is inserted in such a way, that the first significant figure after decimal is at the *2nd* place.

$$x = antilog\ \bar{2}.5321$$
$$= 0.03405$$

Thus $x = 0.03405$

EXERCISE – 6.4

1- Write down the characteristic of the logarithms of the following numbers.

(i) 6350 (ii) 2035.6 (iii) 2.057

(iv) 0.8657 (v) 0.0732 (vi) 0.000721

2- Write down the values of:

(i) $\log 52.13$ (ii) $\log 6.304$ (iii) $\log 0.6127$

(iv) $\log 0.0057$ (v) $\log 0.00003$

3- If $\log 6374 = 3.8044$, write down the values of:

(i) $\log 6.374$ (ii) $\log 0.6374$ (iii) $\log 0.00637$

4- (i) If $\log x = \bar{2}.0374$, find x. (ii) If $\log x = 0.1597$, find x.

(iii) If $\log x = 4.4236$, find x.

6.5 LAWS OF LOGARITHM

(i) $\log_a (mn) = \log_a m + \log_a n$

(ii) $\log_a \left(\frac{m}{n}\right) = \log_a m - \log_a n$

(iii) $\log_a m^n = n \log_a m$

Proof

i- Let m and n are positive integers and 'a' is any admissible base, then taking

$$x = \log_a m \text{ ....(i)}$$
$$y = \log_a n \text{ ....(ii)}$$

Then $a^x = m,\ a^y = n.$

Thus $mn = a^x . a^y = a^{x+y}$

$$\log_a mn = x + y$$
$$= x + y$$

$$\boxed{\log_a mn = \log_a m + \log_a n} \quad \text{from (i) and (ii)}$$

ii- Let m and n are positive integers and 'a' is any admissible base, *(i.e a > 1)*, then

$$x = \log_a m \text{ ...(i)}$$
$$y = a \log_a n \text{ ...(ii)}$$

Therefore $a^x = m,\ a^y = n.$

$$\frac{m}{n} = \frac{a^x}{a^y}$$
$$= a^x . a^{-y}$$
$$= a^{x-y}$$

$$log_a\left(\frac{m}{n}\right) = x - y$$

$$log_a\left(\frac{m}{n}\right) = x - y$$

$$log_a\left(\frac{m}{n}\right) = log_a m - log_a n \quad \textit{from (i) and (ii)}$$

iii- Let $x = log_a m$, then as before

$$a^x = m \text{ and}$$

$$(a^x)^n = m^n$$

$$m^n = (a^x)^n$$

$$= a^{nx}$$

Therefore, $log_a m^n = n x$

$$log_a m^n = n\, log_a m$$

6.6 APPLICATION OF LOGARITHM

The logarithm calculated to the base 10 are called common logarithms. We shall denote $log_{10} m$ by $log\, m$ only.

EXAMPLE-1

Show that:

(i) $3\, log 2 + log\, 5 = log\, 40$

(ii) $log 2 + 2\, log\, 5 - log\, 3 - 2\, log\, 7 = log\left(\frac{50}{147}\right)$

(iii) $log\left(\frac{9}{14}\right) + log\left(\frac{35}{24}\right) - log\left(\frac{15}{16}\right) = 0$

(iv) $7\log\left(\frac{16}{15}\right)+5\log\left(\frac{25}{24}\right)+3\log\left(\frac{81}{80}\right)=\log 2$

SOLUTION: (i) $L.H.S = 3\log 2+\log 5$

$= \log 2^3+\log 5$

$= \log 8+\log 5$

$= \log(8\times 5)$

$= \log 40$

$= R.H.S$

Thus $3\log 2+\log 5=\log 40$

(ii) $L.H.S=\log 2+2\log 5-\log 3+2\log 7$

$= \log 2+\log 5^2-(\log 3+2\log 7)$

$= \log 2+\log 25-(\log 3+\log 7^2)$

$= \log(2\times 25)-(\log 3+\log 49)$

$= \log(50)-(\log 3\times 49)$

$= \log 50-\log 147$

$= \log\left(\frac{50}{147}\right)$

(iii) $L.H.S = \log\left(\frac{9}{14}\right)+\log\left(\frac{35}{24}\right)-\log\left(\frac{15}{16}\right)$

$= \log\left(\frac{9}{14}\times\frac{35}{24}\right)-\log\left(\frac{15}{16}\right)$

$$= \log\left(\frac{3}{2}\times\frac{5}{8}\right)-\log\left(\frac{15}{16}\right)$$

$$= \log\left(\frac{15}{16}\right)-\log\left(\frac{15}{16}\right)$$

$$= 0$$

$$= R.H.S$$

Thus $\log\left(\frac{9}{14}\right)+\log\left(\frac{35}{25}\right)-\log\left(\frac{15}{16}\right)=0$

(iv) $L.H.S = 7\log\left(\frac{16}{15}\right)+5\log\left(\frac{25}{24}\right)+3\log\left(\frac{81}{80}\right)$

$$= \log\left(\frac{16}{15}\right)^7+\log\left(\frac{25}{24}\right)^5+\log\left(\frac{81}{80}\right)^3$$

$$= \log\left[\left(\frac{16}{15}\right)^7\times\left(\frac{25}{24}\right)^5\times\left(\frac{81}{80}\right)^3\right]$$

$$= \log\left[\left(\frac{2^4}{3\times5}\right)^7\times\left(\frac{5^2}{2^3\times3}\right)^5\times\left(\frac{3^4}{2^4\times3}\right)^3\right]$$

$$= \log\left[\frac{2^{28}}{3^7\times5^7}\times\frac{5^{10}}{2^{15}\times3^5}\times\frac{3^{12}}{2^{12}\times5^3}\right]$$

$$= \log\left[2^{28-15-12}\times5^{10-7-3}\times3^{12-7-5}\right]$$

$$= \log\left[2^1\times5^0\times3^0\right]$$

$$= \log[2\times1\times1]$$

$$= \log 2$$

$$= R.H.S$$

EXAMPLE-2

Evaluate: (i) $\frac{log32}{log4}$ (ii) $\frac{log27}{log9}$

SOLUTION: (i) $\frac{log32}{log4} = \frac{log2^5}{log2^2}$

$$= \frac{5\,log2}{2\,log2}$$

$$= \frac{5}{2}$$

Thus $\frac{log32}{log4} = \frac{5}{2}$

(ii) $\frac{log27}{log9} = \frac{log3^3}{log3^2}$

$$= \frac{3\,log3}{2\,log3}$$

$$= \frac{3}{2}$$

Thus $\frac{log27}{log9} = \frac{3}{2}$

EXAMPLE-3

Simplify without using the log table.

(i) $log\ 5 + log\ 6 - log\ 2$

(ii) $log\ 88.44 + log\ 66.76 - log\ 48.55$

(iii) $log\ 7.44 + log\ 5 + log\ 99 - log\ 7$

SOLUTION: (i) $log\ 5 + log\ 6 - log\ 2$

$$= log\,(5\times6) - log\ 2$$

$$= log\left(\frac{5\times6}{2}\right)$$

(ii) $\log 88.44 + \log 66.76 - \log 48.55$

$$= \log(88.44 \times 66.76) - \log 48.55$$

$$= \log\left(\frac{88.44 \times 66.76}{48.55}\right)$$

(iii) $\log 7.44 + \log 5 + \log 99 - \log 7$

$$= \log(7.44 \times 5 \times 99) - \log 7$$

$$= \log\left(\frac{7.44 \times 5 \times 99}{7}\right)$$

EXAMPLE-4

Evaluate using logarithm table. $\frac{25.36 \times 2.4569}{847.5}$.

SOLUTION: Let $x = \frac{25.36 \times 2.4569}{847.5}$

$$\text{Then } \log x = \log\left(\frac{25.36 \times 2.4569}{847.5}\right)$$

$$= \log(25.36 \times 2.4569) - \log(847.5)$$

$$= \log(25.36) + \log(2.4569) - \log(847.5)$$

$$= 1.4041 + 0.3903 - 2.9281$$

$$= -1.1337 = -1 - 0.1337$$

$$= -1 - 1 + 1 - 0.1337$$

$$= -2 + 0.8663$$

$$x = \text{antilog}(\bar{2}.8663)$$

$$x = 0.07351$$

EXAMPLE-5

Evaluate using logarithm table. $\dfrac{8492\times 3.72}{47.82\times 52.24}$

SOLUTION: Let $x = \dfrac{8492\times 3.72}{47.8\times 52.24}$

$$\text{Then} \quad \log x = \log\left(\frac{8492\times 3.72}{47.8\times 52.24}\right)$$

$$= \log(8432\times 3.72) - \log(47.8\times 52.24)$$

$$= \log 8492 + \log 3.72 - (\log 47.8 + \log 52.24)$$

$$= \log 8492 + \log 3.72 - \log 47.8 - \log 52.24$$

$$= 3.9290 + 0.5705 - 1.6794 - 1.7180$$

$$= 4.4995 - 3.3974$$

$$= 1.1021$$

$$x = \text{antilog}(1.1021)$$

$$= 12.65$$

Hence the value of given expression = 12.65.

EXERCISE – 6.5

1- Solve

(i) $\dfrac{\log 81}{\log 9}$ (ii) $\dfrac{\log 36}{\log 6}$ (iii) $\dfrac{\log 243}{\log 9}$

2- Evaluate

(i) $\log 5 + \log 4 + \log 3 - \log 6$

(ii) $\log 5 + \log 20 + \log 24 + \log 25 - \log 60$

(iii) $2\log 3 + 3\log 4 + 4\log 5 - 2\log 6$

(iv) $2\log 5 + \log 8 - \frac{1}{2}\log 4$

(v) $\log 200 + \log 5$

Hint: in each part write $\log 5 = \log\left(\frac{10}{2}\right) = \log 10 - \log 2 = 1 - \log 2$

3- Simplify without using logarithm table.

(i) $\log 1.3472 + \log 22.79 - \log 5$

(ii) $\log 22.13 + \log 0.354 + \log 7 - \log 3$

(iii) $\log 57.86 + \log 4.385 - \log 2.391 - \log 3.072$

4- Solve with the help of logarithm table.

(i) $\dfrac{2.38 \times 3.901}{4.83}$ (ii) $\dfrac{8.67 \times 3.94}{1.78}$

(iii) $\dfrac{25.36 \times 3.4569}{9.87 \times 8.93}$

5- Prove that

(i) $\log\left(\frac{a^2}{bc}\right)+\log\left(\frac{b^2}{ca}\right)+\log\left(\frac{c^2}{ab}\right)=0$

(ii) $3\log 2+2\log 3+\log 5=\log 360$

(iii) $5\log 3-\log 9=\log 27$

(iv) $\log\left(\frac{75}{16}\right)+\log\left(\frac{32}{243}\right)-2\log\left(\frac{5}{9}\right)=\log 2$

(v) $2\log\left(\frac{11}{13}\right)+\log\left(\frac{130}{77}\right)-\log\left(\frac{55}{91}\right)=\log 2$

6- Show that: $3\log 4+2\log 5-\frac{1}{3}\log 64-\frac{1}{2}\log 16=2$

7- Show that: $\log(1\times 2\times 3)=\log 1+\log 2+\log 3$

8- Using logarithmic table evaluate the following:

(i) $69.13\times 0.34\times 0.014$ (ii) $\frac{8.67\times 3.94}{1.78}$ (iii) $\frac{4}{3}\times 3.142\times(1.5)^3$

(iv) $\frac{(25.36)^2\times(0.4569)}{847.5}$ (v) $\frac{0.9876\times(16.42)^2}{(4.567)^{1/3}}$

(vi) $\sqrt{\frac{3\sqrt{0.0125}\times\sqrt{31.15}}{0.00081}}$ (vii) $\frac{(6.45)^3\times(0.00034)^{1/3}\times(981.9)}{(9.37)^2\times(8.93)^{1/4}\times(0.0617)}$

(viii) $\frac{(0.0437)^{2/3}\times(1.407)^2}{(0.0015)^{1/3}\times(1.235)^{1/7}}$

9- If $v=\sqrt{\frac{g\ell}{2\pi}}$ find v. When $\ell=150$, $g=32.16$, $\pi=3.142$

10- If $H=\frac{I^2Rt}{4.2}$ find H. When $I=1.3$, $R=6.7$, and $t=25$

11- Find h, if $h=\frac{v}{\pi(R^2-r^2)}$, when $v=1190$, $R=83.6$, $r=62.4$ and $\pi=3.14$

Review Exercise – 6

1- Encircle the correct answer.

(i) $\sqrt{3}$ is:

(a) *a rational number* (b) *an irrational number*
(c) *a complex number* (d) *an integer*

(ii) $\sqrt[3]{7}$ is called:

(a) *radical* (b) *radicand*
(c) *rational number* (d) *integer*

(iii) In $\sqrt{3}$, 3 is called.

(a) *radical* (b) *radicand*
(c) *integer* (d) *natural number*

(iv) In a^n, n is called

(a) *radical* (b) *radicand*
(c) *exponent* (d) *base*

(v) In 4^5, 4 is called

(a) *base* (b) *exponent*
(c) *integer* (d) *radical*

(vi) The logarithm calculated to the base *"10"* is called

(a) *mantissa* (b) *common logarithm*
(c) *characteristic* (d) *natural number*

(vii) In the logarithm of a number the integral part is called.

(a) *characteristic* (b) *mantissa*
(c) *decimal part* (d) *real part*

(viii) In the logarithm of a number the decimal part is called

(a) *characteristic* (b) *mantissa*
(c) *rational number* (d) *real part*

(ix) $\sqrt{\sqrt{2}} = ?$

(a) 2^2 (b) 2

(c) $2^{1/2}$ (d) $2^{1/4}$

(x) $\sqrt{2+\sqrt{3}}$ is not radical, because $2+\sqrt{3}$ is:

(a) *irrational* (b) *rational*

(c) *integer* (d) *exponent*

2- Fill in the blanks.

(i) If $\sqrt[n]{a}$ is irrational, where "a" is rational, then $\sqrt[n]{a}$ is called ____________ .

(ii) The symbol $\sqrt[n]{\ }$ is called ____________ .

(iii) In 3^5, 5 is called the ____________ .

(iv) In a^n, "a" is called the ____________ .

(v) The logarithm calculated to the base 10 is called ____________ .

(vi) The logarithm of a number consists of two parts, the integral part is called ____________ .

(vii) In the logarithm of a number the decimal part is called ____________ .

3- Simplify:

(i) $(x^5y^3)^{\frac{1}{2}} \times (y^7x^3)^{-\frac{1}{3}}$

(ii) $(a^{\frac{1}{4}}b^{\frac{1}{3}})^{-\frac{1}{2}} \div (a^{\frac{1}{3}}b^{\frac{1}{4}})^{-3}$

4- Evaluate:

(i) $x^{\frac{2}{3}}y^{\frac{5}{8}} \times y^{\frac{1}{2}} \div (xy)^{\frac{1}{3}}$

(ii) $\left(\frac{2}{5}\right)^{-1} \div \left(\frac{4}{25}\right) \times 625$

5- Show that $\log \frac{(3\times4\times5)}{7} = log3 + log4 + log5 - log7$

6- Use logarithmic table to evaluate:

(i) $62.14 \times 0.32 \times 0.015$

(ii) $\frac{3.64 \times 3.94}{2.78}$

(iii) $\frac{(13.26)^2 \times (0.4564)}{325.5}$

SUMMARY

- If $\sqrt[n]{a}$ is irrational, where a is a rational number, then $\sqrt[n]{a}$ is called a radical of order n.

- The symbol $\sqrt[n]{\ }$ is called the radical sign of index n. In $\sqrt[n]{a}$, a is called radicand.

- For any real number "a" and a positive integer "n" we define $a^n = a \times a \times a \times \cdots \times a$ *(n times)*. Here "a" is called the base and "n" the exponent.

- The logarithm calculated to the base *10* is called a common logarithm.

- The logarithm of a number consists of two parts, the integral part is called the characteristic and the decimal part is called the mantissa.

- Scientific notation is a method to write very large number in the form $a = b \times 10^n$.

- A number whose square root is non-negative is called a real number.

UNIT 7

ARITHMETIC AND GEOMETRIC SEQUENCES

- **Sequence**
- **Arithmetic Sequence**
- **Arithmetic Mean**
- **Geometric Sequence**
- **Geometric Mean**

After completion of this unit, the students will be able to:

- Define a sequence (progression) and its terms.
- Know that a sequence can be constructed form a formula or an inductive definition.
- Identify arithmetic sequence.
- Find the nth or the general term of an arithmetic sequence.
- Solve problems involving arithmetic sequence.
- Know arithmetic mean between two numbers.
- Insert n arithmetic means between two numbers.
- Identify a geometric sequence.
- Find the nth or the general term of a geometric sequence.
- Solve problems involving geometric sequence.
- Know geometric mean between two numbers.
- Insert n geometric means between two numbers.

7.1 SEQUENCE (Progression)

In our daily life, we often observe things which increase or decrease progressively by fixed amounts. For example:

1- Number of days pass in a year by *7 days* every week.
2- Our age increases by *12 months* every year.
3- The price of a thing increases by a fixed amount, as you increase the number of units of that thing one-by-one.

In order to study such situations from daily life, let us consider the concept of a sequence. A sequence is an arrangement of numbers written in definite order according to some specific rule. A sequence is also called progression.

Look at the following number patterns.

i- $1,3,5,7,9,\cdots$

ii- $2,4,6,8,10,\cdots$

iii- $1,4,9,16,25,\cdots$

iv- $1,\frac{1}{2},\frac{1}{4},\frac{1}{8},\frac{1}{16},\cdots$

v- $1,\frac{1}{3},\frac{1}{9},\frac{1}{27},\frac{1}{81},\cdots$

From these number patterns, it can be noticed that each successive number, can be found by applying a specific rule that justifies the position of succeeding one. This shows that all the numbers of each pattern are in a definite order.

From *(i)*, the rule is:
Start with *1*, then add *2* to each term to get the next term.

From *(ii)*, the rule is:
Start with *2*, then add *2* to each term to get the next term.

From *(iii)*, the rule is:
Square each number of $1, 2, 3, 4, 5, \cdots$

From *(iv)*, the rule is:
Start with 1, and multiply each term by $\frac{1}{2}$ to get the next term.

From *(i)*, to *(iv)* we say each pattern form a number sequence. The number in a sequence are the terms of the sequence.

To represent a sequence, a special notation a_n is adopted and the symbol $\{a_n\}$ or $a_1, a_2, a_3, \cdots a_n, \cdots$ is used. *(Read the final dots "..." as "and so forth")*.

$$
\begin{aligned}
a_1 &= \text{1st term,} \\
a_2 &= \text{2nd term,} \\
a_3 &= \text{3rd term,} \\
&\cdots\cdots\cdots \\
&\cdots\cdots\cdots \\
a_n &= \text{nth term or general term.}
\end{aligned}
$$

7.1.1 Finite and Infinite Sequences

Look at the following number patterns.

(i) $1, 2, 3, 4, \cdots$	*(ii)* $1, 3, 5, 7, \cdots, 15$
(iii) $1, \frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \cdots$	*(iv)* $2, 4, 6, 8, \cdots, 20$
(v) $1, 4, 7, 10, \cdots$	*(vi)* $1, \frac{1}{3}, \frac{1}{9}, \frac{1}{27}, \cdots$

If there is a last term in a sequence, it is called a finite sequence. In the above examples, *(ii)* and *(iv)* are **finite sequences.**

If there is no last term in a sequence, it is called an **infinite sequence.** In the given examples *(i), (iii), (v), (vi)* are infinite sequences.

7.1.2 Construction of a Sequence from a Formula

Now we write the sequence with the help of *nth* term.

If $a_n = 2n+3, \quad n = 1,2,3,\cdots,8$ then

$a_1 = 2\times 1+3 = 2+3 = 5$

$a_2 = 2\times 2+3 = 4+3 = 7$

$a_3 = 2\times 3+3 = 6+3 = 9$

$a_4 = 2\times 4+3 = 8+3 = 11$

$a_5 = 2\times 5+3 = 10+3 = 13$

$a_6 = 2\times 6+3 = 12+3 = 15$

$a_7 = 2\times 7+3 = 14+3 = 17$

$a_8 = 2\times 8+3 = 16+3 = 19$

The sequence is $5,7,9,11,13,15,17,19.$

The terms of the sequence $\{a_n\}$ have been written by assigning the values $1,2,3,\ldots,8$ to n. For example:

If $a_n = (-1)^{n+1}\,(n+3)$ and $n = 1,2,3,4$ then

$a_1 = (-1)^{1+1}\,(1+3) = (-1)^2\,(4) = 1\times 4 = 4$

$a_2 = (-1)^{2+1}\,(2+3) = (-1)^3\,(5) = -1\times 5 = -5$

$a_3 = (-1)^{3+1}\,(3+3) = (-1)^4\,6 = 1\times 6 = 6$

$a_4 = (-1)^{4+1}\,(4+3) = (-1)^5\,7 = -1\times 7 = -7$

The sequence is : $4, -5, 6, -7.$

With the help of *nth* term, we can write any desired term by giving a particular value to "n".

EXERCISE – 7.1

1- Write the first three terms of the following:

(i) $a_n = n+3$ (ii) $a_n = (-1)^n n^3$ (iii) $a_n = 3n+5$

(iv) $a_n = \dfrac{n+1}{2n+5}$ (v) $a_n = \dfrac{1}{(2n-1)^2}$ (vi) $a_n = n+3 = 2$

(vii) $a_n = \dfrac{1}{3^n}$ (viii) $a_n = 3n-5$ (ix) $a_n = (n+1)a_{n-1}, a_1 = 1$

2- Find the terms indicated in the following sequences.

(i) $2,6,11,17,\cdots,a_8$ (ii) $1,3,12,60,\cdots,a_7$ (iii) $1,\dfrac{1}{3},\dfrac{1}{9},\dfrac{1}{27},\cdots,a_6$

(iv) $-1,1,3,5,\cdots,a_9$ (v) $\dfrac{1}{3},\dfrac{2}{5},\cdots,a_5$ (vi) $1,-3,5,-7,\cdots,a_9$

3- Find the next four terms of the following sequences.

(i) $12,16,21,27,\cdots$ (ii) $1,3,7,15,31,\cdots$ (iii) $-1,2,12,40,\cdots$

(iv) $9,11,14,17,19,22,\cdots$ (v) $4,8,12,16,\cdots$ (vi) $-2,0,2,4,6,8,10,\cdots$

7.2 ARITHMETIC SEQUENCE *(Progression)*

An arithmetic progression *(abbreviated A.P)* is a sequence of numbers called terms, each of which after the first is obtained from the preceding one by adding to it a fixed number called "common difference" of the progression.

Let 'a' be the first term and 'd' be the common difference in an *A.P*. Then the second term is $a+d$, the *3rd* term is $a+2d$. In each of these terms the co-efficient of d is one less than the number of the term. Similarly the *10th* term is $a+9d$.

The *nth* term is *(n – 1)th* after the *1st* term and thus is obtained after 'd' has been added $(n-1)$ times, then

General term = *nth* term $= a_n = a+(n-1)d.$

If we take $n = 12$ then *12th* term $= a_{12} = a+(12-1)d = a+11d$

EXAMPLE-1

Find the general term and the 14th term of an A.P, whose 1st term is 2 and the common difference is 5.

SOLUTION: Given $a_1 = a = 2,\ d = 5$, we know that:

$$\begin{aligned} a_n &= a+(n-1)d \\ &= 2+(n-1)5 \\ &= 2+5n-5 \\ &= 2-5+5n \\ &= 5n-3 \end{aligned}$$

General term = *nth* term $= a_n = 5n-3$

Now putting $n = 14$ in equation $a_n = a + (n-1)d$, we have

$$\begin{aligned} a_{14} &= a+(14-1)d \\ &= 2+13\times5 \\ &= 2+65 \\ &= 67 \end{aligned}$$

$$a_{14} = 67$$

EXAMPLE-2

If 5^{th} term of an A.P is 16 and 20^{th} term is 46, what is the 15^{th} term ?

SOLUTION: Given $a_5 = 16$ and $a_{20} = 46$

Since $a_n = a+(n-1)d$ .... (1)

Putting $n = 5$ in equation (1), we have

$$a_5 = a+(5-1)d$$

$$16 = a+4d \text{ .... (2)}$$

Putting $n = 20$ in equation (1), we have

$$a_{20} = a+(20-1)d$$

$$46 = a+19d \text{ .... (3)}$$

Subtracting equation (2) from (3), we have

$$46-16 = a-a+19d-4d$$

$$30 = 15d \Rightarrow \boxed{d = 2}$$

Putting $d = 2$ in equation (2), we have

$$16 = a+4\times 2$$

$$16 = a+8 \Rightarrow 16-8 = a$$

$$\Rightarrow \boxed{a = 8}$$

Putting $n = 15$ in equation (1), we have

$$a_{15} = a+(15-1)d$$

$$= 8+14\times 2$$

$$= 8+28$$

$$= 36$$

Thus $\boxed{a_{15} = 36}$

EXAMPLE-3

Find the number of terms in an A.P,

if $a_1 = 3,\ d = 4,\ a_n = 59$

SOLUTION: Given $a_1 = a = 3,\ d = 4,\ a_n = 59$

Since $a_n = a+(n-1)d$

$$59 = 3+(n-1)4$$

$$= 3+4n-4$$

$$= 4n-1$$

$$60 = 4n$$

$$n = \frac{60}{4} = 15$$

$$\boxed{n = 15}$$

Thus the number of terms in the given A.P is 15.

EXAMPLE-4

If $a_{n-3} = 2n - 12$, find a_n.

SOLUTION: Given $a_{n-3} = 2n - 12$

Putting $n = n+3$, we have,

$$a_{n+3-3} = 2(n+3) - 12$$

$$a_n = 2n + 6 - 12$$

$$= 2n - 6$$

$$\boxed{a_n = 2n - 6}$$

EXERCISE – 7.2

1- Find the specified term of the following A.P

(i) $3, 7, 11, \cdots$, 61st term. (ii) $-4, -7, -10, \cdots, a_{19}$

(iii) $6, 4, 2, \cdots$, 45th term. (iv) $9, 14, 19, \cdots, a_{14}$

(v) $11, 6, 1, \cdots, a_{18}$

2- Find the missing element using the formula of A.P $a_n = a + (n-1)d$

(i) $a = 2, a_n = 402, n = 26$, (ii) $a_n = 81, d = -3, n = 18$

(iii) $a = 5, a_n = 61, n = 15$ (iv) $a = 16, a_n = 0, d = -\frac{1}{4}$

(v) $a = 10, a_n = 400, d = 5$ (vi) $a_n = 261, d = 4, n = 18$

3- Find the 15^{th} term of an A.P where the 3^{rd} term is 8 and the common difference is $\frac{1}{3}$.

4- Which term of an A.P $6, 2, -2, \cdots$ is -146 ?

5- Which term of an A.P $5, 2, -1, \cdots$ is -118 ?

6- How many terms are there in an A.P, in which $a_1 = a = 11, a_n = 68, d = 3$.

7- Find the 11^{th} term of an A.P $2-x, 3-2x, 4-3x, \cdots$

8- Find the n^{th} term of an A.P, where $a_{n-5} = 3n + 9$.

9- Find the n^{th} term of an A.P: $\left(\frac{3}{4}\right)^2, \left(\frac{3}{7}\right)^2, \left(\frac{3}{10}\right)^2, \cdots$

10- If the n^{th} term of an A.P is $3n - 5$. Find the A.P.

7.3 ARITHMETIC MEAN

A number 'A' is said to be an arithmetic mean between the two numbers 'a' and 'b', if a, A, b is an $A.P.$

$$A - a = b - A \quad \text{(Common Difference)}$$
$$A + A = a + b$$
$$2A = a + b$$
$$A = \frac{a+b}{2}$$

EXAMPLE-1

Find $A.M$ between 4 and 8.

SOLUTION: Given $a = 4, b = 8$

$$A = \frac{a+b}{2}$$
$$= \frac{4+8}{2}$$
$$= \frac{12}{2} = 6$$
$$A = 6$$

$A.M$ represents Arithmetic Mean

EXAMPLE-2

Find an $A.M$ between $2\sqrt{5}$ and $6\sqrt{5}$.

SOLUTION: Given $a = 2\sqrt{5},\ b = 6\sqrt{5}$

$$A = \frac{a+b}{2}$$
$$= \frac{2\sqrt{5}+6\sqrt{5}}{2}$$
$$= \frac{8\sqrt{5}}{2}$$
$$= 4\sqrt{5}$$

7.3.2 Arithmetic Means Between Two Numbers

Let $A_1, A_2, A_3 \cdots A_n$, be "n" $A.Ms$ between the two numbers a and b, such that $a, A_1, A_2, A_3 \cdots, A_n, b$ is an $A.P.$

Here $a_1 = a$, $a_{n+2} = b$, because there are $n + 2$ terms in an $A.P.$

Using $a_n = a+(n-1)d$, we have

$$a_{n+2} = a+(n+2-1)d$$

$$b = a+(n+1)d$$

$$b-a = (n+1)d$$

$$\frac{(b-a)}{n+1} = d \quad \text{or} \quad \boxed{d = \frac{b-a}{n+1}}$$

$$A_1 = a+d = a+\frac{b-a}{n+1} = \frac{an+a+b-a}{n+1} = \frac{na+b}{n+1}$$

$$A_2 = a+2d = a+2\left(\frac{b-a}{n+1}\right) = \frac{na+a+2b-2a}{n+1} = \frac{na+a+2b}{n+1} = \frac{(n-1)a+2b}{n+1}$$

$$A_3 = a+3d = a+3\left(\frac{b-a}{n+1}\right) = \frac{na+a+3b-3a}{n+1} = \frac{na-2a+3b}{n+1} = \frac{(n-2)a+3b}{n+1}, \cdots,$$

$$A_n = a+nd = a+n\left(\frac{b-a}{n+1}\right) = \frac{na+a+nb-na}{n+1} = \frac{a+nb}{n+1}$$

EXAMPLE-1

If 8 and 12 are two A.Ms between a and b. Find a and b.

SOLUTION: $a, 8, 12, b$ is an $A.P$

Common difference $= d$

$= a_3 - a_2$

$= 12 - 8 = 4$

and $b = a_4$

$$= a_3 + d$$

$$b = 12 + 4 = 16$$

$$a = a_2 - d$$

$$= 8 - 4 = 4$$

Thus $a = 4, b = 16$

EXAMPLE-2

Find three A.Ms between $\sqrt{3}$ and $9\sqrt{3}$.

SOLUTION: Let A_1, A_2, A_3 be three A.Ms between $\sqrt{3}$ and $9\sqrt{3}$

such that $\sqrt{3}, A_1, A_2, A_3, 9\sqrt{3}$ is an A.P.

Here $a_1 = a = \sqrt{3},\ n = 5,\ a_5 = 9\sqrt{3}$

Using $a_n = a + (n-1)d$

$$a_5 = a + (5-1)d$$

$$9\sqrt{3} = a + 4d$$

$$9\sqrt{3} = \sqrt{3} + 4d$$

$$9\sqrt{3} - \sqrt{3} = 4d$$

$$4d = 8\sqrt{3}$$

$$d = 2\sqrt{3}$$

Thus $A_1 = a + d = \sqrt{3} + 2\sqrt{3} = 3\sqrt{3}$

$$A_2 = A_1 + d = 3\sqrt{3} + 2\sqrt{3} = 5\sqrt{3}$$

$$A_3 = A_2 + d = 5\sqrt{3} + 2\sqrt{3} = 7\sqrt{3}$$

Thus $3\sqrt{3}, 5\sqrt{3}, 7\sqrt{3}$ are the required three A.Ms between $\sqrt{3}$ and $9\sqrt{3}$.

EXERCISE – 7.3

1- Find $A.M$ between:

(i) $-3, 7$ (ii) $x-1, x+7$

(iii) $\sqrt{7}, 3\sqrt{7}$ (iv) x^2+x+1 ; x^2-x+1

2- If 3 and 6 are two $A.Ms$ between a and b, find a and b.

3- Find three $A.Ms$ between 11 and 19.

4- Find three $A.Ms$ between $\sqrt{2}$ and $6\sqrt{2}$

5- Find six $A.Ms$ between 5 and 8.

6- Find seven $A.Ms$ between 8 and 12.

7- If the $A.M$ between 5 and b is 10, then find the value of b.

8- If the $A.M$ between a and 10 is 40, then find the value of "a".

9- If the three $A.Ms$ between a and b are $5, 9$ and 13, find a and b.

7.4 GEOMETRIC SEQUENCE (Progression)

A geometric progression (abbreviated $G.P$) is a sequence of numbers called terms, each of which after the first is obtained by multiplying the preceding one by a fixed number called common ratio. This common ratio is denoted by 'r' which cannot be zero in any case. We can obtain common ratio as:

$$r = \frac{a_2}{a_1} = \frac{a_3}{a_2} = \frac{a_4}{a_3} = \cdots = \frac{a_n}{a_{n-1}} = \cdots$$

Let a be the first term and r be the common ratio in a $G.P$, then the second term is ar. Third term is ar^2. In each term the exponent of r is one less than the number of the term. Similarly the eighth term is ar^7 and nth term is ar^{n-1}. Thus the general term of $G.P$ is $a_n = ar^{n-1}$

EXAMPLE-1

Find the 5th term of a G.P, in which $a = 2$, $r = 3$.

SOLUTION: Given $a = 2$, $r = 3$, $n = 5$, $a_5 = ?$

Since $a_n = ar^{n-1}$

$$a_5 = ar^{5-1}$$
$$a_5 = 2(3)^4$$
$$= 2 \times 81$$
$$= 162$$
$$\boxed{a_5 = 162}$$

EXAMPLE-2

If $a_4 = \frac{8}{27}$, $a_7 = \frac{-64}{729}$ in G.P. Find a_{10} .

SOLUTION: Given $a_4 = \frac{8}{27}$, $a_7 = \frac{-64}{729}$,

Here we will find a and r first.

Since $a_n = ar^{n-1}$

$$a_4 = ar^{4-1}$$
$$\frac{8}{27} = ar^3$$
$$ar^3 = \frac{8}{27} \quad \text{........(i)}$$

and $a_7 = ar^{7-1}$

$$\frac{-64}{729} = ar^6$$
$$ar^6 = -\frac{64}{729} \quad \text{........(ii)}$$

Now dividing (ii) by (i), we have, $\frac{ar^6}{ar^3} = \frac{\frac{-64}{729}}{\frac{8}{27}}$

$$r^3 = \frac{-64 \times 27}{729 \times 8} = -\frac{8}{27} = \left(-\frac{2}{3}\right)^3$$

$$r = -\frac{2}{3}, \textit{ putting in (i)}$$

$$a\left(-\frac{2}{3}\right)^3 = \frac{8}{27} \Rightarrow a\left(-\frac{2}{3}\right)^3 = \left(\frac{2}{3}\right)^3$$

$$a = -1$$

Then $a_{10} = ar^{10-1}$

$$= (-1)\left(\frac{2}{3}\right)^9 = -\left(\frac{2}{3}\right)^9$$

EXERCISE – 7.4

1- Find the 7^{th} term of a G.P $2, 8, 32, \cdots$

2- Find the 11^{th} term of a G.P $2, 6, 18, \cdots$

3- Find the 6^{th} term of a G.P $-\frac{3}{2}, 3, -6, \cdots$

4- Find the 5^{th} term of a G.P $4, -12, 36, \cdots$

5- Find the missing elements of the G.P:

(i) $r = 10$, $a_n = 100$, $a = 1$ (ii) $a_n = 400$, $r = 2$, $a = 25$

(iii) $a = 128$, $r = \frac{1}{2}$, $a_n = \frac{1}{4}$

6- Find the 11^{th} term of a G.P whose 5^{th} term is 9 and common ratio is 2.

7- Find the 13^{th} term of a G.P whose 7^{th} term is 25 and common ratio is 3.

8- If a, b, c, d are in G.P, show that, $a - b$, $b - c$, $c - d$, are in G.P.

9- Find the n^{th} term of a G.P, if $\frac{a_5}{a_3} = \frac{4}{9}$ and $a_2 = \frac{4}{9}$.

10- Find three consecutive numbers in G.P, whose sum is 26 and their product is 216.

11- Find the 30th term of a G.P $x, 1, \frac{1}{x}, \cdots$

12- Find the pth term of a G.P $x, x^3, x^5, \cdots$

7.5 GEOMETRIC MEAN (G.M)

A number 'G' is said to be a geometric mean between the two numbers a and b, if a, G, b is a geometric progression.

$$\frac{G}{a} = \frac{b}{G} \text{ (common ratio)}$$

$$G^2 = ab$$

$$G = \pm\sqrt{ab}$$

Positive G.M between a and b is $\sqrt{ab}$.

EXAMPLE-1

Find the G.M between 3 and 27.

SOLUTION: Given $a = 3,\ b = 27$. then

$$G = \pm\sqrt{ab}$$
$$= \pm\sqrt{3\times 27}$$
$$= \pm\sqrt{81}$$
$$= \pm 9$$

EXAMPLE-2

Find the G.M between $2x^2$ and $8y^4$.

SOLUTION: Given $a = 2x^2\ ,\ b = 8y^4$

$$G = \pm\sqrt{ab}$$
$$= \pm\sqrt{2x^2\times 8y^4}$$
$$= \pm\sqrt{16x^2y^4}$$
$$= \pm\sqrt{(4xy^2)^2}$$
$$= \pm 4xy^2$$

7.5.1 'n' Geometric Means Between Two Numbers

Let $G_1, G_2, G_3 \cdots, G_n$ be the n G.Ms between the two numbers a and b such that $a, G_1, G_2, G_3 \cdots, G_n, b$ is a G.P, there are $n + 2$ terms in this G.P. in which $a_1 = a\ ,\ a_{n+2} = b$,

Using $a_n = ar^{n-1}$

$$a_{n+2} = ar^{n+2-1}$$
$$b = ar^{n+1}$$
$$r^{n+1} = \frac{b}{a}$$
$$r = \left(\frac{b}{a}\right)^{\frac{1}{n+1}}$$

$$G_1 = a\times r = a\times\left(\frac{b}{a}\right)^{\frac{1}{n+1}}$$

$$G_2 = G_1\times r = ar^2 = a\left(\frac{b}{a}\right)^{\frac{2}{n+1}}$$

$$G_3 = G_2\times r = ar^3 = a\left(\frac{b}{a}\right)^{\frac{3}{n+1}}$$

$$\vdots \quad \vdots \quad \vdots \quad \vdots$$

$$G_n = G_{n-1}\times r = ar^n = a\times\left(\frac{b}{a}\right)^{\frac{n}{n+1}}$$

EXAMPLE-1

Insert two G.Ms between 4 and $\frac{1}{2}$.

SOLUTION: Let G_1, G_2 be the two G.Ms between 4 and $\frac{1}{2}$ such that:

$4, G_1, G_2, \frac{1}{2}$ is a G.P,

Here $a = 4$, $n = 4$, $a_4 = \frac{1}{2}$,

Since $a_n = ar^{n-1}$

$$a_4 = ar^{4-1}$$

$$\frac{1}{2} = ar^3$$

$$ar^3 = \frac{1}{2}$$

$$4r^3 = \frac{1}{2}$$

$$r^3 = \frac{1}{8} = \frac{1}{2^3}$$

$$r^3 = \left(\frac{1}{2}\right)^3$$

$$r = \frac{1}{2}$$

Thus $G_1 = a\times r = 4\times\frac{1}{2} = 2$

$$G_2 = G_1\times r = 2\times\frac{1}{2} = 1$$

EXERCISE – 7.5

1- Find *G.M* between: (i) 9 and 5 (ii) 4 and 9 (iii) -2 and -8.

2- Insert two *G.Ms* between: (i) 1 and 8 (ii) 3 and 81

3- Insert three *G.Ms* between: (i) 1 and 16 (ii) 2 and 32

4- Insert four real geometric means between 3 and 96.

5- The *A.M* between two numbers is 5 and their positive *G.M* is 4. Find the numbers.

6- The positive *G.M* between two numbers is 6 and the *A.M* between them is 10. Find the numbers.

7- Show that the *A.M* between the two numbers 4 and 8 is greater than their geometric mean.

8- Insert four geometric means between 160 and 5.

9- Insert three geometric means between 486 and 6.

10- Insert four geometric means between $\frac{1}{8}$ and 128.

11- Insert six geometric means between 56 and $-\frac{7}{16}$.

12- Insert five geometric means between $\frac{32}{81}$ and $\frac{9}{2}$.

Review Exercise – 7

1- Encircle the correct answer.

(i) Third term of $a_n = n+3$, when $n = 0$ is

(a) 3 (b) 6 (c) 9 (d) 0

(ii) Fourth term of $a_n = \frac{1}{(2n-1)^2}$, when $n = 0$ is

(a) $\frac{1}{7}$ (b) $\frac{1}{49}$ (c) $\frac{1}{81}$ (d) 0

(iii) For $2, 6, 11, 17, \cdots, a_5$ is

(a) 24 (b) 30 (c) 21 (d) 22

(iv) Next term of $12, 16, 21, 27$ is

(a) 34 (b) 30 (c) 31 (d) 32

(v) a_6 of $3, 7, 11, \cdots$ is

(a) 3 (b) 19 (c) 23 (d) 20

(vi) A.M between $\sqrt{3}$ and $3\sqrt{3}$ is

(a) $2\sqrt{3}$ (b) $5\sqrt{3}$ (c) $9\sqrt{3}$ (d) $4\sqrt{3}$

(vii) A.M between $2\sqrt{5}$ and $6\sqrt{5}$ is

(a) $4\sqrt{5}$ (b) $3\sqrt{5}$ (c) $5\sqrt{5}$ (d) $7\sqrt{5}$

(viii) a_5 of $2, 6, 18, \cdots$ is

(a) 160 (b) 161 (c) 162 (d) 30

(ix) G.M between -3 and -12 is

(a) ± 6 (b) ± 9 (c) ± 36 (d) ± 3

(x) G.M between 1 and 8 is

(a) $2\sqrt{2}$ (b) $\pm 2\sqrt{2}$ (c) $-2\sqrt{2}$ (d) $\sqrt{2}$

2- Fill in the blanks.

(i) The general or nth term of a sequence is denoted by____________

(ii) If $a_n = 2n + 3$, then a = ____________

(iii) In an $A.P$ $a_n = a + (n-1)d$, is called ____________

(iv) $A.M$ between 5 and 15 is ____________

(v) If a, A, b is an $A.P$ then A = ____________

(vi) In a $G.P.$ "r" is called____________

(vii) In a $G.P.$ a_n = ____________

(viii) If a, G, b is a $G.P$, then G = ____________

(ix) Positive geometric mean between 2 and 8 is ____________

(x) The nth term of an $A.P$ when $a_{n-5} = 3n + 9$ ____________

3- Find the general term and the $18th$ term of an $A.P$, whose first term is 3 and the common difference is 2.

4- Find the nth term of an $A.P$ $\left(\frac{3}{5}\right)^3, \left(\frac{3}{7}\right)^3, \left(\frac{3}{9}\right)^3, \cdots$

5- If the $A.M$ between a and 16 is 24. Then find the value of 'a'.

6- Find the $15th$ term of a $G.P.$ whose $7th$ term is 27 and common ratio is 3.

7- Insert four Geometric Means between $\frac{1}{2}$ and 16.

8- Find the three consecutive number in $G.P$, whose sum is 26 and their product is 216.

SUMMARY

- In a number pattern each successive number can be found by applying a specific rule that justifies the position of succeeding one, that is all the members in a pattern are in a definite order, such a number pattern is called a sequence.

- A sequence in which each term is obtained from the previous term by adding a fixed number is called an arithmetic sequence.

- A number "A" is said to be an arithmetic mean between the two numbers a and b if a, A, b is arithmetic sequence.

- A sequence in which each term is obtained from the previous term by multiplying it with a common ratio is called a geometric sequence.

- A number "G" is said to be a geometric mean between the two numbers a and b if a, G, b is a geometric sequence.

UNIT 8

SETS AND FUNCTIONS

- Operations on Sets
- Binary Relation
- Function

After completion of this unit, the students will be able to:

- Recall the sets denoted by N,Z,W,E,O,P and Q.
- Recognize set operations ($\cup, \cap, \backslash, \ldots$).
- Perform the following operations on sets:
  - Union,
  - Intersection,
  - Complement.
- Verify the following fundamental properties of union and intersection of two or three given sets.
  - Commutative property of union and intersection.
  - Associative property of union and intersection.
- Use Venn diagram to represent
  - Union and intersection of sets,
  - Complement of a set.
- Use Venn diagram to verify
  - Commutative laws for union and intersection of sets.
  - Associative laws for union and intersection of sets.
  - De Morgan's laws.
- Define binary relation and identify its domain and range.
- Define function and identify its domain and range.
- Demonstrate the following
  - Into function,
  - One-one function,
  - Into and one-one function (injective function),
  - Onto function (surjective function),
  - One-one and onto function (bijective function).

8.1 SET

Every thing in the universe whether living or non-living is called an object. We give name to particular type of collection of objects such as *"hockey team"*,*"herd of cattle"* *"bunch of flowers"* etc.

A set means a collection of well-defined objects i.e. the collection of objects is given in such a way that it is possible to tell without doubt, whether the given object belongs to the collection or not.

A collection of well defined distinct objects is called a *"Set"*.

Sets are usually denoted by capital alphabets $A, B, C, \cdots, X, Y, Z$.

The objects in a set are called its members or elements. Elements are denoted by small letters or numbers.

For example:

(i) $A = \{1,3,4,5\}$
(ii) $B = \{a,e,i,o,u\}$
(iii) $C = \{1,3,5,7,9, \cdots\}$ are all sets.

If an object x belongs to a set *"A"*, we write it as $x \in A$, it means that x is an elements of set *"A"*.
If an object *"x"* does not belong to a set A, we write it as $x \notin A$.

IMPORTANT SETS

Set of Natural Number

Counting numbers are called natural numbers, for example $1,2,3$ and so on. Thus set of natural numbers denoted by N is:

$$N = \{1,2,3,4,5, \cdots\}$$

Set of Whole Numbers

The set of whole numbers denoted by *'W'* is :

$$W = \{0,1,2,3,4,5, \cdots\}$$

Set of Integers

The set of integers denoted by '$\mathbb{Z}$' is :

$$\mathbb{Z} = \{\cdots, -3, -2, -1, 0, 1, 2, 3, \cdots\}$$

Set of Even Numbers

The set of even number denoted by 'E' is :

$$E = \{\cdots, -4, -2, 0, 2, 4, 6, 8, \cdots\}$$

Set of Odd Numbers

The set of odd number denoted by 'O' is :

$$O = \{\cdots, -3, -1, 1, 3, 5, \cdots\}$$

Set of Prime Numbers

The set of prime numbers denoted by 'P' is :

$$P = \{2, 3, 5, 7, 11, 13, 17, \cdots\}$$

Prime numbers means the number, which is divisible by *1* and itself.

Set of Rational Numbers

The set of rational numbers denoted by 'Q' is :

$$Q = \{\frac{p}{q} : q \neq 0,\ p,\ q \in \mathbb{Z}\}$$

8.1.1 Operations on Sets

Like operations, addition, subtraction, multiplication and division on numbers in Arithmetic, there are certain operations on sets like union, intersection and complement etc.

Union of Sets

If A and B are the two non-empty sets, then the union of A and B means the set of all those elements which are either present in A or in B or in both; It is denoted by $A \cup B$.

Thus $A \cup B = \{x : x \in A \text{ or } x \in B\}$

For example : Let $A = \{1,2,3,4,5\}$ and

$B = \{2,4,6,8,10\}$ then

$A \cup B = \{1,2,3,4,5\} \cup \{2,4,6,8,10\}$

$= \{1,2,3,4,5,6,8,10\}$

EXAMPLES

(i) Find $A \cup B$, *if*

$A = \{a,b,c\}$ $B = \{a,e,i,o,u\}$

SOLUTION: Given $A = \{a,b,c\}$ $B = \{a,e,i,o,u\}$

Then $A \cup B = \{a,b,c\} \cup \{a,e,i,o,u\}$

$= \{a,b,c,e,i,o,u\}$

(ii) Find $C \cup D$, *if*

$C = \{2,3,4,5\}$ $D = \{6,7\}$

SOLUTION: Given $C = \{2,3,4,5\}$ $D = \{6,7\}$

Then $C \cup D = \{2,3,4,5\} \cup \{6,7\}$

$= \{2,3,4,5,6,7\}$

(iii) Find $E \cup F$, *if*

$E = \{1,2,3,5,7\}$ $F = \{2,4,6,8\}$

SOLUTION: Given $E = \{1,2,3,5,7\}$ $F = \{2,4,6,8\}$

Then $E \cup F = \{1,2,3,5,7\} \cup \{2,4,6,8\}$

$= \{1,2,3,4,5,6,7,8\}$

Intersection of Sets

The intersection of two sets A and B denoted by $A \cap B$ is the set of all those elements which are common to both A and B.

Thus $A \cap B = \{x : x \in A \wedge x \in B\}$

EXAMPLE

Find $A \cap B$, if

(i) $A = \{2,3,5,7,11\}$, $B = \{1,3,5,7,9\}$

(ii) $A = \{3,6,9,12,15,18,21,24\}$, $B = \{4,8,12,16,20,24,28,32\}$

(iii) $A = \{1,2,3,4,6,12\}$, $B = \{1,2,3,4,6,9,18\}$

SOLUTION:

(i) Given $A = \{2,3,5,7,11\}$, $B = \{1,3,5,7,9\}$

$$A \cap B = \{2,3,5,7,11\} \cap \{1,3,5,7,9\}$$

$$= \{3,5,7\}$$

(ii) Given $A = \{3,6,9,12,15,18,21,24\}$, $B = \{4,8,12,16,20,24,28,32\}$

$$A \cap B = \{3,6,9,12,15,18,21,24\} \cap \{4,8,12,16,20,24,28,32\}$$

$$= \{12,24\}$$

(iii) Given $A = \{1,2,3,4,6,12\}$, $B = \{1,2,3,4,6,9,18\}$

$$A \cap B = \{1,2,3,4,6,12\} \cap \{1,2,3,4,6,9,18\}$$

$$= \{1,2,3,4,6\}$$

Universal Set

If there are some sets under consideration, there happens to be a set, which is a super set of each one of the given sets, such a set is called the universal set. It is denoted by 'U'.

For example:

If $A = \{1,2\}$, $B = \{2,3\}$, $C = \{1,3\}$, then $U = \{1,2,3\}$, which is superset of the given sets A, B, C and A, B, C are called subsets of U.

Complement of a Set

Let A be a sub-set of a universal set U, then the complement of A with respect to U, denoted by A' or $U - A$ or A^c is the set of all those elements of U, which are not in A.

Thus $A' = \{x \mid x \in U \wedge x \notin A\}$

$x \in A' \Rightarrow x \notin A$

and $x \in A \Rightarrow x \notin A'$

> ***Remember that:*** $A' = U - A$
>
> $U' = U - U = \phi,\ \phi' = U - \phi = U$
>
> $(A')' = U - A' = A$

EXAMPLE-1

If $U = \{1,2,3,4,5,6,7\}$ *and* $A = \{3,4,5\}$ *find* A'.

SOLUTION:

Given $U = \{1,2,3,4,5,6,7\}$

$A = \{3,4,5\}$, *then*

$A' = U - A$

$= \{1,2,3,4,5,6,7\} - \{3,4,5\}$

Thus $A' = \{1,2,6,7\}$

EXAMPLE-2

If $U = \{1,2,3,4,5,6,7,8\}$ *and* $A = \{2,3,5,7\}$ *find* $A', A \cup A', A \cap A'$

SOLUTION:

Given $U = \{1,2,3,4,5,6,7,8\}$ *and*

$A = \{2,3,5,7\}$

Then $A' = U - A$

$= \{1,2,3,4,5,6,7,8\} - \{2,3,5,7\}$

$= \{1,4,6,8\}$

we note that: $A \cup A' = \{2,3,5,7\} \cup \{1,4,6,8\}$

$= \{1,2,3,4,5,6,7,8\}$

$= U$

and $A \cap A' = \{2,3,5,7\} \cap \{1,4,6,8\}$

$= \{\ \}$

$= \Phi$

i.e. $A \cup A' = U$ and $A \cap A' = \Phi$

EXAMPLE-3

If $U = \{1,2,3,\ldots,20\}$, $B = \{9,10,11,12,\ldots,20\}$ then find $B', B \cup B', B \cap B,$

SOLUTION: $B' = U - B$

$= \{1,2,3,\ldots,20\} = \{9,10,11,12,\ldots,20\}$

$= \{1,2,3,4,5,6,7,8\}$

Now $B \cup B' = \{9,10,11,12,\ldots,20\} \cup \{1,2,3,4,5,6,7,8\}$

$= \{1,2,3,4,5,6,7,8,9\ldots,20\}$

$= U$

and $B \cap B' = \{9,10,11,12,\ldots,20\} \cap \{1,2,3,4,5,6,7,8\}$

$= \{\ \}$

$= \Phi$

i.e. $B \cup B' = U$ and $B \cap B' = \Phi$

Difference of two sets

If A and B are two non-empty sets, then A difference B is the set of all the elements of set A which are not present in B, symbolically it is written as $A-B$ or $A \backslash B$. Similarly B difference A is the set of all those elements of the B which are not present in A. For example:

(i) Let $A = \{2,3,4,5\}$, $B = \{2,4,6,8\}$

Then $A - B = \{2,3,4,5\} - \{2,4,6,8\} = \{3,5\}$

and $B - A = \{2,4,6,8\} - \{2,3,4,5\} = \{6,8\}$

$A - B \neq B - A$

(ii) Let $A = \{3,4,5,6,7\}$, $B = \{1,2,3,4,7,8,9,10\}$

Then $A \backslash B = \{3,4,5,6,7\} \backslash \{1,2,3,4,7,8,9,10\} = \{5,6\}$

and $B \backslash A = \{1,2,3,4,7,8,9,10\} \backslash \{3,4,5,6,7\}$

$= \{1,2,8,9,10\}$

8.1.2 Properties of Union of Sets

Commutative Law

For any two sets A and B

$$A \cup B = B \cup A$$

Proof: Let $A \cup B = \{x : x \in A \text{ or } x \in B\}$

$= \{x : x \in B \text{ or } x \in A\}$

$= B \cup A$

Hence $A \cup B = B \cup A$

Associative Law

For any three sets A, B and C,

$$(A \cup B) \cup C = A \cup (B \cup C)$$

Proof: Let $(A \cup B) \cup C = \{x : x \in (A \cup B) \text{ or } x \in C\}$

$= \{x : (x \in A \text{ or } x \in B) \text{ or } x \in C\}$

$= \{x : x \in A \text{ or } (x \in B \text{ or } x \in C)\}$

$= \{x : x \in A \text{ or } x \in (B \cup C)\}$

$= A \cup (B \cup C)$

Hence $(A \cup B) \cup C = A \cup (B \cup C)$

EXAMPLE

If $A = \{1,2,3,4\}$, $B = \{7,8,9,10\}$ and $C = \{2,6\}$

Prove that (a) $A \cup B = B \cup A$

(b) $(A \cup B) \cup C = A \cup (B \cup C)$

SOLUTION: *Given* $A = \{1,2,3,4\}$, $B = \{7,8,9,10\}$, $C = \{2,6\}$ then

(a) $A \cup B = \{1,2,3,4\} \cup \{7,8,9,10\}$

$= \{1,2,3,4,7,8,9,10\}$........... (i)

$B \cup A = \{7,8,9,10\} \cup \{1,2,3,4\}$

$= \{1,2,3,4,7,8,9,10\}$........... (ii)

From (i) and (ii) $A \cup B = B \cup A$

(b) $(A \cup B) \cup C = (\{1,2,3,4\} \cup \{7,8,9,10\}) \cup \{2,6\}$

$= \{1,2,3,4,7,8,9,10\} \cup \{2,6\}$

$= \{1,2,3,4,6,7,8,9,10\}$..........(i)

$A \cup (B \cup C) = \{1,2,3,4\} \cup (\{7,8,9,10\} \cup \{2,6\})$

$= \{1,2,3,4\} \cup \{2,6,7,8,9,10\}$

$= \{1,2,3,4,6,7,8,9,10\}$..........(ii)

From (i) and (ii) $(A \cup B) \cup C = A \cup (B \cup C)$

Properties of Intersection of Sets

Commutative Law

For any two sets A and B,

$$A \cap B = B \cap A$$

Proof: Let $A \cap B = \{x : x \in A \text{ and } x \in B\}$

$= \{x : x \in B \text{ and } x \in A\}$

$= \{x : x \in (B \cap A)\}$

$= B \cap A$

Thus $A \cap B = B \cap A$

Associative Law

For any three sets A, B and C,

$$(A \cap B) \cap C = A \cap (B \cap C)$$

Proof: Let $(A \cap B) \cap C$

$= \{x : x \in (A \cap B) \text{ and } x \in C\}$

$= \{x : (x \in A \text{ and } x \in B) \text{ and } x \in C\}$

$= \{x : x \in A \text{ and } x \in B \text{ and } x \in C\}$

$= \{x : x \in A \text{ and } (x \in B \text{ and } x \in C)\}$

$= \{x : x \in A \text{ and } x \in (B \cap C)\}$

$= \{x : x \in A \cap (B \cap C)\}$

$= A \cap (B \cap C)$

Thus $(A \cap B) \cap C = A \cap (B \cap C)$

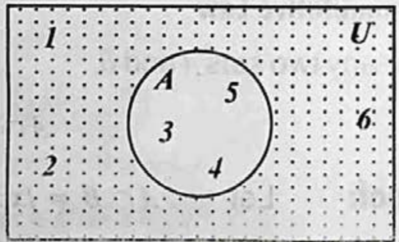
8.1.3 Venn Diagram

To express a relationship among sets in a perspective way, we represent the sets by means of diagrams, known as Venn diagram. They were first used by an English logician and the mathematician John Venn *(1834 to 1883 A.D)*.

In a venn-diagram, the universal set is usually represented by a rectangular region and its subsets are represented by closed bounded regions inside this rectangular region.

For example: $U = \{1,2,3,4,5,6,\}$ and $A = \{3,4,5\}$,

The rectangular region shown in the figure represents the universal set U and the region enclosed by a closed circle inside the rectangular region represents the set A.



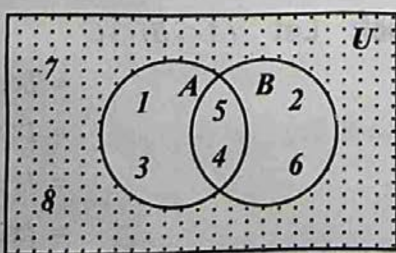
The dotted region of U outside A represents complement of A. i.e. A' thus $A' = \{1,2,6\}$

Union and Intersection of Sets

Let $U = \{1, 2, 3, 4, 5, 6, 7, 8\}$ and

$A = \{1, 3, 4, 5\}$ and $B = \{2, 4, 5, 6\}$ be its two sub-sets.

In the figure the rectangular region represents U *(universal set)*. Since A and B are intersecting sets, we draw two intersecting circles representing A and B respectively, then:

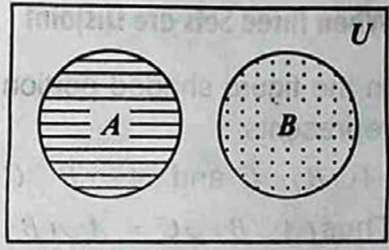


(1) The total region bounded by A and B represents $A \cup B$, therefore, $A \cup B = \{1, 3, 4, 5\} \cup \{2, 4, 5, 6\} = \{1, 2, 3, 4, 5, 6\}$

(2) The common region between the two sets A and B represents $A \cap B$ therefore, $A \cap B = \{1, 3, 4, 5\} \cap \{2, 4, 5, 6\} = \{4, 5\}$

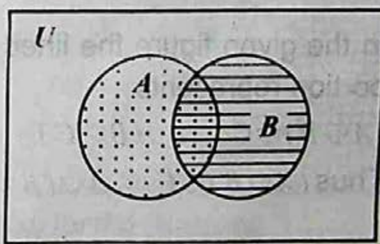
8.1.3.4 Commutative Laws for Union and Intersection of Sets by Venn Diagram.

When two Sets are Disjoint



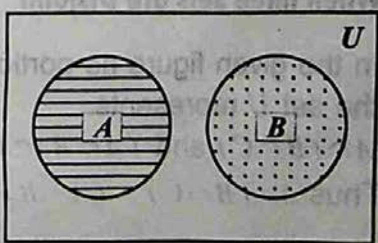
In the given figure, the lined portion and the dotted portion represents $A \cup B$. Also the dotted portion and the lined portion represents $B \cup A$. This means that the same portion represents $B \cup A$ and $A \cup B$, thus $A \cup B = B \cup A$

When two Sets are Overlapping



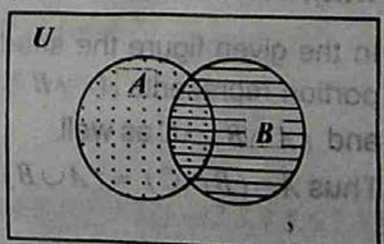
In the given figure, the dotted portion, dotted and lined portion and the lined portion represent, $A \cup B$. Also the lined portion, lined and dotted portion and the dotted portion represent $B \cup A$.
Thus $A \cup B = B \cup A$.

When two Sets are Disjoint



In the given figure, the lined portion represents the set A, whereas the dotted portion represents the set B. That is no part of U represents $A \cap B$ and $B \cap A$. Thus $A \cap B = B \cap A$.

When two Sets are Overlapping

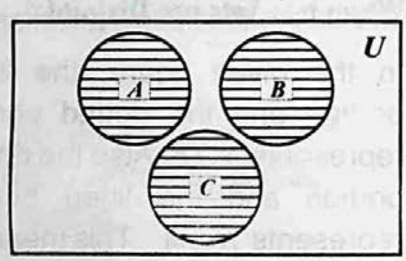


In the given figure, the dotted and the lined and dotted portion represents A, whereas the lined and the dotted and the lined portion represents B. The lined and dotted portion is common to both the sets which represents $A \cap B$ and $B \cap A$. Thus $A \cap B = B \cap A$.

Associative Laws for Union and Intersection of Sets by Venn Diagram.

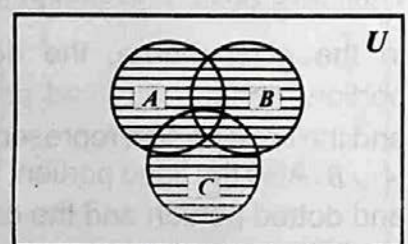
When three Sets are Disjoint

In the figure shaded portion represents $(A \cup B) \cup C$ and $A \cup (B \cup C)$.
Thus $(A \cup B) \cup C = A \cup (B \cup C)$



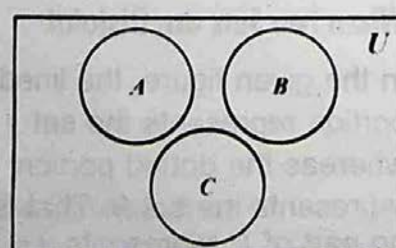
When three Sets are Overlapping

In the given figure the lined portion represents $(A \cup B) \cup C$, $A \cup (B \cup C)$.
Thus $(A \cup B) \cup C = A \cup (B \cup C)$.



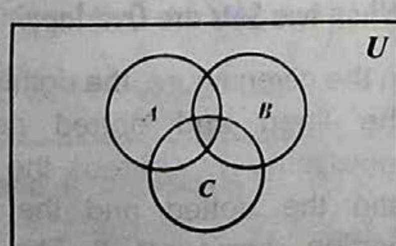
When three Sets are Disjoint

In the given figure no portion of the set U represents $A \cap (B \cap C)$ and $(A \cap B) \cap C$.
Thus $A \cap (B \cap C) = (A \cap B) \cap C$.



When three Sets are Intersecting

In the given figure the shaded portion represents $A \cap (B \cap C)$ and $(A \cap B) \cap C$ as well.
Thus $A \cap (B \cap C) = (A \cup B) \cap C$.



De Morgan's Laws

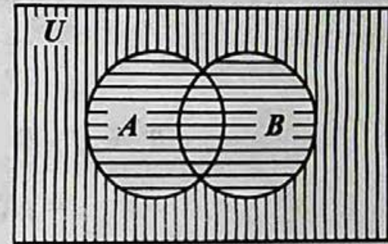
When A and B are any two sub-sets of a universal set U, then

(i) $(A \cup B)' = A' \cap B'$

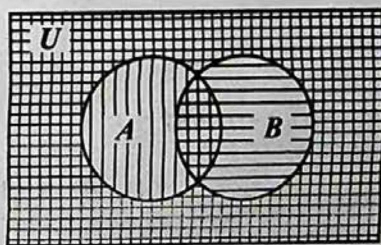
or $(A \cup B)^c = A^c \cap B^c$

(ii) $(A \cap B)' = A' \cup B'$

or $(A \cap B)^c = A^c \cup B^c$



In the given figure $A \cup B$ is represented by ≡ lines, where as $(A \cup B)'$ is represented by |||| lines.



In this figure checked '#' region represents $A' \cap B'$. Therefore in two figures we see that the '||||' region and the region '#' are same therefore $(A \cup B)' = A' \cap B'$

The second part of this is left as an exercise for the students.

EXAMPLE

If $U = \{1,2,3,4,5,6,7,8,9,10,11,12\}$,

$A = \{2,3,4,6,8,9,10\}$,

and $B = \{2,3,5,7,9\}$ *then*

verify De Morgan's Law.

SOLUTION: *Given* $A = \{2,3,4,6,7,8,9,10\}$

$B = \{2,3,5,7,9\}$

(a) $A \cup B = \{2,3,4,6,7,8,9,10\} \cup \{2,3,5,7,9\}$

$= \{2,3,4,5,6,7,8,9,10\}$

$(A \cup B)^c = U - (A \cup B)$

$= \{1,2,3,4,5,6,7,8,9,10,11,12\} - \{2,3,4,5,6,7,8,9,10\}$

$= \{1,11,12\}$......(i)

$A^c = U - A$

$= \{1,2,3,4,5,6,7,8,9,10,11,12\} - \{2,3,4,5,6,7,8,9,10\}$

$= \{1,11,12\}$

$B^c = U - B$

$= \{1,2,3,4,5,6,7,8,9,10,11,12\} - \{2,3,5,7,9\}$

$= \{1,4,6,8,10,11,12\}$

$A^c \cap B^c = \{1,11,12\} \cap \{1,4,6,8,10,11,12\}$

$= \{1,11,12\}$......*(ii)*

From (i) and (ii)

$(A \cup B)^c = A^c \cup B^c$

(b) $A \cap B = \{2,3,4,6,7,8,9,10\} \cap \{2,3,5,7,9\}$

$= \{2,3,7,9\}$

$(A \cap B)^c = U - (A \cap B)$

$= \{1,2,3,4,5,6,7,8,9,10,11,12\} - \{2,3,7,9\}$

$= \{1,4,5,6,8,10,11,12\}$......*(i)*

$A^c = U - A$

$= \{1,5,11,12\}$

$B^c = U - B$

$= \{1,4,6,8,10,11,12\}$

$A^c \cup B^c = \{1,4,6,8,10,11,12\}$......*(ii)*

From (i) and (ii)

$(A \cap B)^c = A^c \cup B^c$

EXERCISE – 8.1

1- If $A = \{1,4,7,8\}$, $B = \{4,6,8,9\}$ and $C = \{3,4,5,7\}$ Find:

(i) $A \cup B$ (ii) $B \cup C$

(iii) $A \cap C$ (iv) $A \cap (B \cap C)$

(v) $(A \cup B) \cup C$ (vi) $(A \cap B) \cap C$

2- If $A = \{1,7,11,15,17,21\}$, $B = \{11,17,19,23\}$ and $C = \{2,3,5\}$.
Verify that: $(A \cap B) \cap C = A \cap (B \cap C)$

3- If $A = \{2,4,6\}$, $B = \{3,6,9,12\}$ and $C = \{4,6,8,10\}$.
Verify that: $A \cup (B \cup C) = (A \cup B) \cup C$

4- If $A = \{2,3,5,7,9\}$, $B = \{1,3,5,7\}$ and $C = \{2,3,4,5,6\}$.
Verify that: $(A \cap B) \cap C = A \cap (B \cap C)$

5- If $U = \{7,8,9,10,11,12,13,14\}$,
$A = \{7,10,13,14\}$
and $B = \{7,8,11,12\}$ then
verify $(A \cap B)^c = A^c \cup B^c$

6- If $U = \{4,6,8,9,10\}$,
$A = \{4,6\}$
and $B = \{6,8,9\}$ then
verify De Morgan's Law.

7- If $U = \{1,2,3,4,5,6,7,8,9\}$
$A = \{2,3,6,9\}$
and $B = \{1,3,6,7,8\}$ then
verify $(A \cup B)^c = A^c \cap B^c$

8- Fill in the blanks:

(i) $A \cup A =$ ________ (ii) $A \cap A =$ ________

(iii) $A \cup \Phi =$ ________ (iv) $A \cap \Phi =$ ________

(v) $\Phi \cup \Phi =$ ________ (vi) $(A \cap B)' =$ ________

(vii) $(A \cup B)' =$ ________ (viii) $(A')' =$ ________

(ix) $\Phi \cap \Phi' =$ ________ (x) $A \cap A' =$ ________

8.2 BINARY RELATION

Consider the two non-empty sets $A = \{1,2\}$ and $B = \{3,4\}$, then $A \times B = \{(1,3),(1,4),(2,3),(2,4)\}$, where $A \times B$ is called cartesian product from A to B. The elements $(1,3)$, $(1,4)$, $(2,3)$ and $(2,4)$ of $A \times B$ are called ordered pairs.

Similarly $B \times A = \{(3,1),(3,2),(4,1),(4,2)\}$, is a cartesian product from B to A. In general $A \times B \neq B \times A$. All the following sub-sets of $A \times B$ are called binary relations from A to B.

$R_1 = \{\ \}$, $R_2 = \{(1,3)\}$, $R_3 = \{(1,4)\}$

$R_4 = \{(2,3)\}$, $R_5 = \{(2,4)\}$, $R_6 = \{(1,3),(1,4)\}$

$R_7 = \{(1,3),(2,3)\}$, $R_8 = \{(1,3),(2,4)\}$, $R_9 = \{(1,4),(2,3)\}$

$R_{10} = \{(1,4),(2,4)\}$, $R_{11} = \{(2,3),(2,4)\}$, $R_{12} = \{(1,3),(1,4),(2,3)\}$

$R_{13} = \{(1,3),(2,3),(2,4)\}$, $R_{14} = \{(1,4),(2,3),(2,4)\}$,

$R_{15} = \{(1,3),(1,4),(2,4)\}$, $R_{16} = \{(1,3),(1,4),(2,3),(2,4)\}$.

The set of the first elements of the ordered pairs of a binary relation is called its domain.

The set of the second elements of the ordered pairs of a binary relation is called range. Now consider the following examples:

$A = \{1,2,3\}$, $B = \{3,4\}$,

$A \times B = \{(1,3),(1,4),(2,3),(2,4),(3,3),(3,4)\}$

we take a sub-set of $A \times B$ as

$R_1 = \{(1,3),(2,4),(3,4)\}$

R_1 is called a relation or binary relation.

Domain of $R_1 = \{1,2,3\}$
Range of $R_1 = \{3,4\}$

Also if $A = \{4,5,6\}$, then

$A \times A = \{(4,4),(4,5),(4,6),(5,4),(5,5),(5,6),(6,4),(6,5),(6,6)\}$

We take a relation 'R' as any sub-set of $A \times A$ such that:

Domain of $R = \{4,5,6\}$
Range of $R = \{4,5,6\}$

EXAMPLE

If $C = \{1,2\}$, *then write the number of binary relations in* $C \times C$

SOLUTION:

Given $C = \{1,2\}$

Then $C \times C = \{(1,1),(1,2),(2,1),(2,2)\}$

The number of binary relations in $C \times C$ *is equal to* 2^4 *or* 16.

8.3 FUNCTION

Any binary relation "f" between two non-empty sets A and B such that:

(i) Domain $f = A$

(ii) There should be no repetition in the first element of ordered pairs contained in f.

Then "f" is called a function from A to B and expressed as $f: A \rightarrow B$.

EXAMPLE

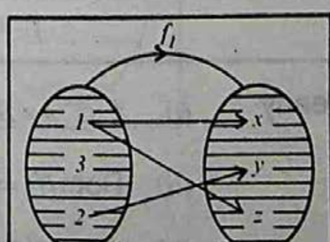
$A = \{1,2,3\}$, $B = \{x,y,z\}$

$A \times B = \{(1,x), (2,x), (3,x), (1,y), (2,y), (3,y), (1,z), (2,z), (3,z)\}$

Consider the following two binary relations.

$f_1 = \{(1,x),(2,y),(1,z)\}$

$f_2 = \{(1,y),(2,x),(3,y)\}$



Binary relation f_1 :

(i) $f_1 \subset A \times B$.

(ii) Dom $f_1 \neq A$

(iii) In 'f_1' the first elements of ordered pairs $(1,x)$ and $(1,z)$. are repeated .

$\therefore$ f_1 is a binary relation but not a function.

Range $(f_1) = \{x, y, z\}$

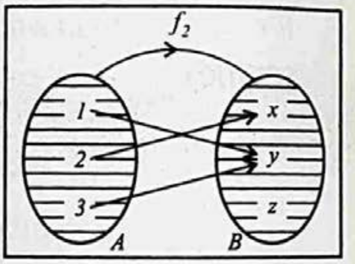
Binary function f_2 :

(i) $f_2 \subset A \times B$.

(ii) Dom $f_2 = \{1, 2, 3\} = A$

(iii) First place elements in ordered pairs of f_2 are not repeated.

$\therefore$ f_2 is a function from A to B.



Into Function Range $(f_2) = \{x, y\} \subset B$ but Range $(f_2) \neq B$

If "f" is a function from A to B such that

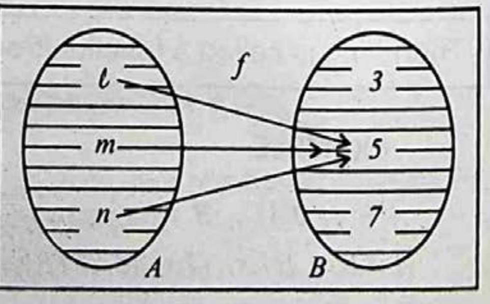
Range $(f) \subset B$ and Range $(f) \neq B$

Then "f" is called A into B function.

For example:

Let $A = \{l, m, n\}$, $B = \{3, 5, 7\}$

$A \times B = \{(l,3),(l,5),(l,7),$
$(m,3),(m,5),(m,7),$
$(n,3),(n,5),(n,7)\}$



Consider a binary relation "f" given by

$$f = \{(l,5),(m,5),(n,5)\}$$

Clearly *(i)* $f \subset A \times B$

(ii) Dom $(f) = \{l, m, n\} = A$

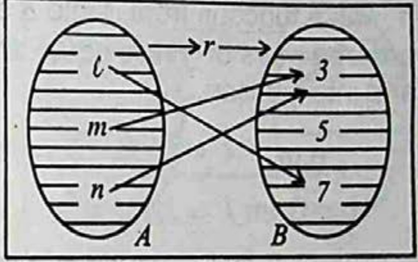
(iii) First elements of any two ordered pairs of "f" are not repeated or in other words, every element of set A is mapped to only one element of set B.

(iv) Range $(f) = \{5\} \subset B$ but Range $(f) \neq B$

Therefore, 'f' is A into B function.

We consider another relation 'r' given by

$$r = \{(\ell,7), (m,3), (n,3)\}.$$



(i) $r \subset A \times B$.

(ii) Dom $(r) = \{\ell, m, n\} = A$

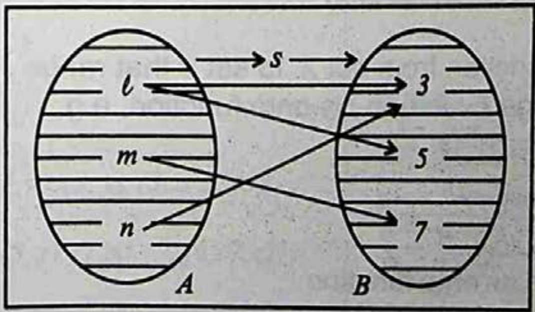
(iii) Every element of set A is associated with one and only one element of set B i.e. to say no repetition in the first element of any two ordered pairs of r takes place.

(iv) Range $(r) = \{3,7\} \neq B$

$\therefore$ r is A into B function.

Let us consider another binary relation s from A to B.

$$s = \{(\ell,3), (\ell,5), (m,7), (n,3)\}.$$



(i) $s \subset A \times B$.

(ii) Dom $(s) = \{\ell, m, n\} = A$.

(iii) There is a repetition of first element 'ℓ' in ordered pairs $(\ell,3)$, $(\ell,5) \in s$ i.e. $\ell \in A$ has been associated with two elements 3 and $5 \in B$.

Therefore, 's' is not a function.

Into and *(one – one)* Function *(Injective Function)*

If f is a function from A into B such that no second elements of any two ordered pairs of f are equal, then it is called an injective i.e. *(one – one)* and into function,

e.g. $A = \{2,3\}$, $B = \{a,b,c\}$, $f = \{(2,a),(3,b)\}$

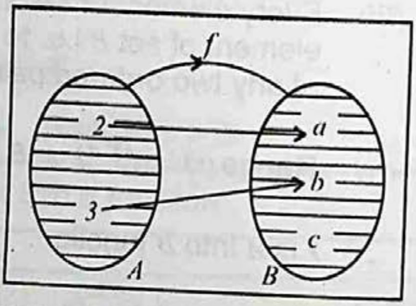
(i) Dom $f = \{2,3\} = A$

(ii) No first place members in ordered pairs of f are repeated.

(iii) Range $(f) \neq B$

$\therefore$ f is A into B function.

(iv) No second place elements in ordered pairs of f are equal (repeated).



Therefore f is an injective function.

Onto Function *(Surjective Function)*

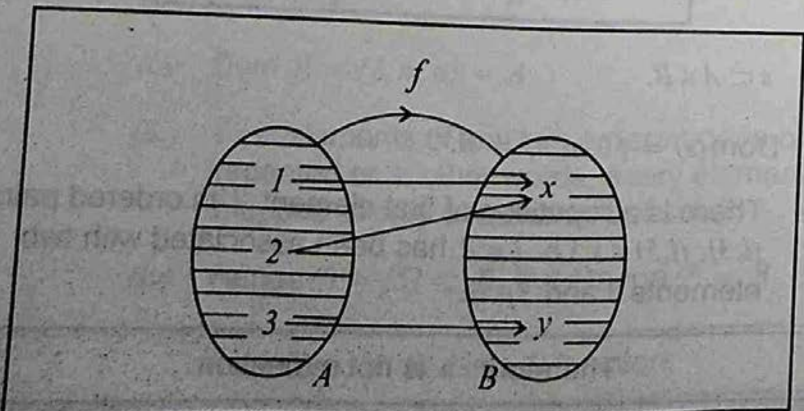
If f is such a function from set A to set B that range f is same as that of set B i.e. range $f = B$ then f is onto function, e.g.

$A = \{1,2,3\}$, $B = \{x,y\}$

$f = \{(1,x),(2,x),(3,y)\}$

Range $f = \{x,y\} = B$

Therefore f is an onto function.



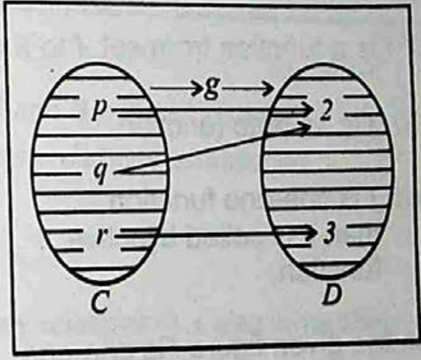
Example:

Let $C = \{p, q, r\}$

$D = \{2, 3\}$

$C \times D = \{(p,2), (q,2), (r,2), (p,3), (q,3), (r,3)\}$

$g = \{(p,2), (q,2), (r,3)\}$



(i) Dom $(g) = \{p, q, r\}$

(ii) No first place elements of any two ordered pairs of g are repeated. Therefore, g is C into D function
Range $(g) = \{2,3\}$.

(iii) Range $(g) = D$

Hence, g is a C onto D function.

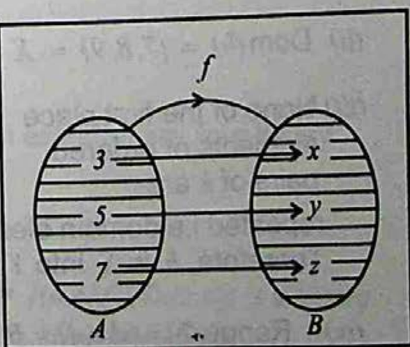
Example:

Let $A = \{3, 5, 7\}$

$B = \{x, y, z\}$

$A \times B = \{(3,x), (3,y), (3,z), (5,x), (5,y), (5,z), (7,x), (7,y), (7,z)\}$.

$f = \{(3,x), (5,y), (7,z)\}$



(i) Dom $(f) = \{3, 5, 7\} = A$

(ii) No first place elements of any two ordered pairs of f are repeated. Therefore, f is A into B function.

(iii) Range $(f) = B$

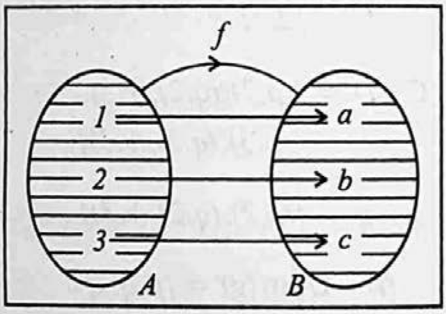
Hence, f is A onto B function.

One-One and onto Function *(Bijective Function)*

If f is a function from set A to B such that:

(i) f is an onto function

(ii) f is one-one function, then it is called bijective function.



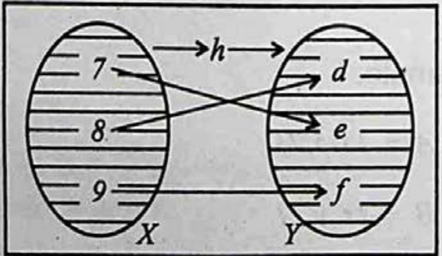
In the given figure f is one-one and onto function for binary relation $f = \{(1,a),(2,b),(3,c)\}$

EXAMPLE

Let $X = \{7, 8, 9\}$, $Y = \{d, e, f\}$

$X \times Y = \{(7,d),(8,d),(9,d),(7,e),(8,e),(9,e),(7,f),(8,f),(9,f)\}$

$h = \{(7,e),(8,d),(9,f)\}$



(i) $h \subset X \times Y$

(ii) Dom $(h) = \{7,8,9\} = X$

(iii) None of the first place elements of ordered pairs of h are repeated i.e domain elements are not repeated. Therefore, h is X into Y function.

(iv) Range $(h) = \{d,e,f\} = B$

Therefore, h is X onto Y function.

(v) No element at the second place of ordered pairs in Y are repeated.

Hence, h is *(one – one)* function.

Therefore, h is a bijective function

EXERCISE – 8.2

1- If $A = \{3,5,6\}$, $B = \{1,3\}$, Find $A \times B$ and $B \times A$ and also the domains and ranges of the two binary relations established at our own for each case.

2- If $A = \{-2,1,4\}$, then write two binary relations in A also write their domains and ranges.

3- Write the number of binary relations possible in each of following cases.

(i) In $C \times C$ when the number of elements in C is 3.

(ii) In $A \times B$ if the number of elements in set A is 3 and in set B is 4.

4- If $L = \{1,2,3\}$, and $M = \{2,3,4\}$,then write a binary relation R such that $R = \{(x,y)/x \in L, y \in M \wedge y \leq x\}$.
Also write Dom(R) and Range (R).

5- If $X = \{0,3,5\}$ and $Y = \{2,4,8\}$, then establish any four binary relations in $X \times Y$.

6- If $A = \{a,b,c\}$ and $B = \{2,4,6\}$ and $f = \{(a,4),(b,4),(c,4)\}$ is a binary relation from $A \times B$, then show that 'f' is A into B function.

7- If $A = \{l,m,n\}$ and $B = \{1,2,3\}$ and $g = \{(l,3),(m,1),(n,1)\}$ is a binary relation from $A \times B$, then show that 'g' is A into B function.

8- If $A = \{1,3,5\}$ and $B = \{x,y,z\}$ and $g = \{(1,x),(3,y),(5,z)\}$ is a binary relation from $A \times B$, then show that 'g' is A onto B function.

Review Exercise – 8

1- Encircle the correct answer.

(i) If A and B are two non-empty sets, then $A \cup B = ?$

(a) Φ (b) $B \cup A$ (c) $A \cap B$ (d) $B \cap A$

(ii) If A and B are two non-empty overlapping sets, then $A \cap B = ?$

(a) Φ (b) $B \cap A$ (c) $A \cap B$ (d) $B \cup A$

(iii) For any two sets A and B, $A \cup B = B \cup A$ is called:

(a) *commutative law* (b) *associative law*
(c) *De-morgan's law* (d) *complement of two sets*

(iv) $A \cup (B \cup C) = (A \cup B) \cup C$ is called

(a) *commutative law* (b) *associative law*
(c) *De-morgan's law* (d) *intersection of sets*

(v) If $U = \{1,2,3,4\}$, $A = \{4\}$, then $A' = ?$

(a) $\{1,2,3\}$ (b) Φ (c) $\{1\}$ (d) $\{1,2,3,4\}$

(vi) If $U = \{1,2,3\}$, $A = \{1\}$, then $U - A = ?$

(a) $\{2,3\}$ (b) $\{1,2\}$ (c) $\{1,3\}$ (d) Φ

(vii) $(A \cup B)' = ?$

(a) $A' \cup B'$ (b) $A' \cap B'$ (c) $(A \cap B)'$ (d) Φ

(viii) $(A \cap B)' = ?$

(a) $A' \cap B'$ (b) $A' \cup B'$ (c) $A \cap B$ (d) $A \cup B$

(ix) If $R = \{(4,5),(5,4),(5,6),(6,4)\}$ then domain of R.

(a) $\{4,6\}$ (b) $\{4,5\}$ (c) $\{4,5,6\}$ (d) $\{5,6\}$

(x) If $R = \{(4,5),(5,4),(5,6),(6,4)\}$ then range of R is:

(a) $\{4\}$ (b) $\{5\}$ (c) $\{6\}$ (d) $\{4,5,6\}$

2- Fill in the blanks.

(i) $(A \cup B)' =$ ______________

(ii) $(A \cap B)' = ?$ ______________

(iii) $A \cup (B \cup C) =$ ______________

(iv) $A \cap (B \cap C) =$ ______________

(v) If A and B be the two non-empty sets, then $A \cup B = B \cup A$ is called the ______________.

(vi) If A and B be the two non-empty sets, then $A \cap B = B \cap A$ is called the ______________.

(vii) Any sub-set of a cartesian product is called a ______________.

(viii) If $R_1 = \{(1,2),(3,4),(5,6)\}$, then domain of R_1 is ______________.

(ix) If $R_1 = \{(1,2),(3,4),(5,6)\}$, then range of R_1 is ______________.

(x) If $f : A \rightarrow B$, then every element of a set A has its image in______________.

3- If $A=\{1,2,3,4,5,6\}$, $B=\{2,3,4,6\}$ and $C=\{2,3,4,7,8,9\}$. Verify that : $(A \cap B) \cap C = A \cap (B \cap C)$

4- If $A=\{2,3,4\}$, $B=\{3,6,9,12\}$ and $C=\{4,6,8,10\}$. Verify that : $A \cup (B \cup C) = (A \cap B) \cup C$

5- If $A=\{2,3,4\}$ and $B=\{1,3\}$. Find $A \times B$ and $B \times A$. Also establish two binary relations each from these cartesian products.

6- Write the number of binary relations possible in each of the following cases.

(i) In $C \times C$, when the number of elements in C are 4.

(ii) In $A \times B$, if number of elements in A are 2 and in B are 3.

7- If $R = \{(a,b) a,b \in W, 3a + 2b = 16\}$. Find its domain and range R.

SUMMARY

- A collection of well-defined distinct objects is called a set.

  $N = \{1, 2, 3, \cdots\}$, is called set of natural numbers.

  $W = \{0, 1, 2, 3, \cdots\}$, is called set of whole numbers.

  $\mathbb{Z} = \{\cdots -1, 0, 1, 2, 3, \cdots\}$, is called set of integers.

  $Q = \{\frac{p}{q} |, p, q \in \mathbb{Z}, q \neq 0\}$, is called set of rational numbers.

  $Q' = A$ set of irrational numbers.

  $R = Q \cup Q' = A$ set of real numbers.

- If there are some sets under consideration there happens to be a set, which is a superset of each one of the given sets, such a set is called the universal set denoted by U.

- Let A be a subset of a universal set U. Then compliment of A denoted by A' or A^c or $U–A$ is the set of all those element of U which are not in A is called complement of a set.

- Let A and B any two sets, then any subset of the cartesian product $A \times B$ is called a binary relation from A to B.

- Any binary relation 'f' between two non-empty sets A and B such that:

  (i) Dom $f = A$

  (ii) First elements in any two of the ordered pairs of f are not repeated, then f is called a function from A to B.

- The union of A and B means the set of all those elements which are either present in A or in B or in both, it is denoted by $A \cup B$.

- The intersection of two sets A and B denoted by $A \cap B$ is the set of all those elements which are common to both A and B.

- If A and B are two non-empty sets, then A difference B is the set of all the elements of set A which are not present in B, symbolically it is written as $A–B$.

UNIT 9

LINEAR GRAPHS

- **Cartesian Plane and Linear Graphs**
- **Conversion Graph**

After completion of this unit, the students will be able to:

- Identify pair of real numbers as an ordered pair.
- Recognize and ordered pair through different examples for instance, an ordered pair (2,3) to represent a seat, located in an examination hall, at the 2nd row and 3rd column.
- Describe rectangular or Cartesian plane consisting of two number lines intersecting at right angles at the point O.
- Identify origin (O) and coordinate axes (horizontal and vertical axes) in the rectangular plane.
- Locate an ordered pair *(a,b)* as a point in the rectangular plane and recognize:
  - a as the x-coordinate (or abscissa),
  - b as the y-coordinate (or ordinate),
- Draw different geometrical shapes (e.g., line segment, triangle and rectangle, etc.) by joining a set of given points.
- Construct a table for pairs of values satisfying a linear equation in two variables.
- Plot the pairs of points to obtain the graph of a given expression.
- Choose an appropriate scale to draw a graph.
- Draw the graph of
  - An equation of the form $y = c$.
  - An equation of the form $x = a$.
  - An equation of the form $y = mx$.
  - An equation of the form $y = mx + c$.
- Draw a graph from a given table of (discrete) values.
- Identify through graph the domain and the range of function.
- Interpret conversion graph as a linear graph relating to two quantities which are in direct proportion.
- Read a given graph to know one quantity corresponding to another.
- Read the graph for conversions of the form:
  - Miles and kilometres,
  - Acres and hectares,
  - Degrees Celsius and degrees Fahrenheit,
  - Pakistani currency and another currency, etc.

9.1 CARTESIAN PLANE AND LINEAR GRAPHS

9.1.1 Pair of Real Numbers as an Ordered Pair

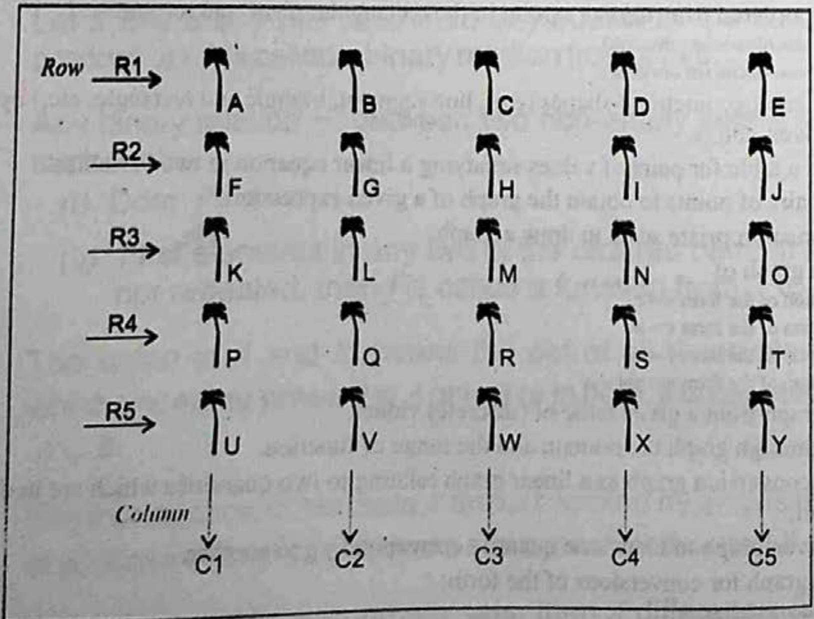
By the definition of equality of sets, for any two elements "a" and "b", we have $\{a,b\} = \{b,a\}$.

However, if we keep in mind the order in which the elements are being listed, then a pair of two elements, listed in a specific order, is called an ordered pair, denoted by (a,b). Thus for different elements "a" and "b", we have $(a,b) \neq (b,a)$.

In general $(a_1,b_1) = (a_2,b_2) \Leftrightarrow a_1 = a_2$ and $b_1 = b_2$ we represent each point in a plane by means of an ordered pair i.e. (x,y).

9.1.2 Ordered Pairs

A gardener prepares an arrangement plan of trees in a square field. The trees are marked with numbers. To identify each tree more easily, the gardener can connect each tree with the column and the row in which it is present. The tree "H" is present in *2nd* row and *3rd* column, while the tree number "R" is present in *4th* row and *3rd* column.



The gardener can write a pair of numbers against the number of a tree in the field as follows.

A(1,1),	B(1,2),	C(1,3),	D(1,4),	E(1,5)
F(2,1),	G(2,2),	H(2,3),	I(2,4),	J(2,5)
K(3,1),	L(3,2),	M(3,3),	N(3,4),	O(3,5)
P(4,1),	Q(4,2),	R(4,3),	S(4,4),	T(4,5)
U(5,1),	V(5,2),	W(5,3),	X(5,4),	Y(5,5)

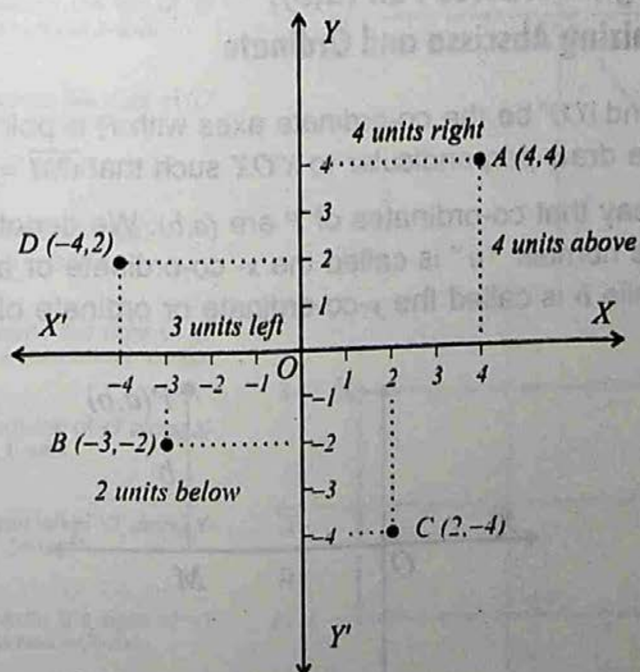
Can we write the pairs of numbers corresponding to the trees *F, G, H, K* and *M*?

Yes these are : *F(2,1), G(2,2), H(2,3), I(2,4), K(3,1)* and *M(3,3)*.

The pairs of numbers *(2,1), (2,2), (2,3), (2,4), (3,1), (3,3)* and so on are examples of **ordered pairs.**

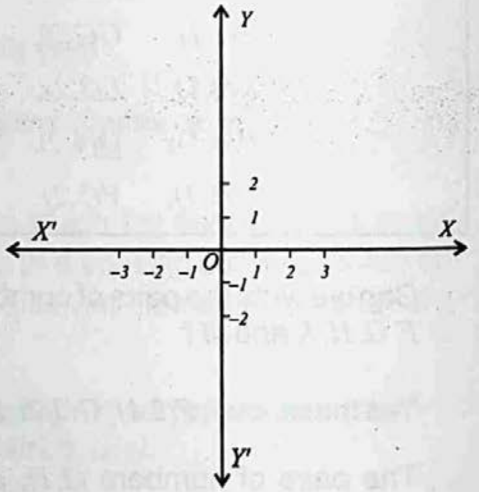
9.1.3 Rectangular or Cartesian Plane

The given figure shows a rectangular or cartesian plane consisting of two number lines *XOX'* and *YOY'* intersecting at right angle at *O*.



9.1.4 Identifying Origin *(O)* and Co-ordinate Axis in Rectangular Plane

In cartesian plane. $X'OX$ and YOY' are two mutually perpendicular lines called co-ordinate axes, intersecting at a point "O". We call the point O as origin.



The horizontal line $X'OX$ is called the *x-axis* and the vertical line YOY' is called the *y-axis*.

We fix a convenient unit of length and mark the origin O.

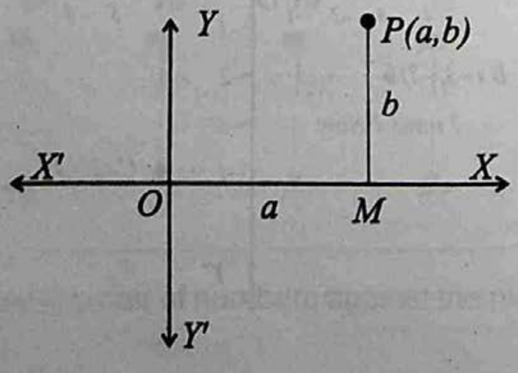
Now we mark equal distances (each equal to unit length) on *x-axis* as well as on *y-axis* on both sides of O.

The distances measured along $\overrightarrow{OX}$ and $\overrightarrow{OY}$ are taken as positive while those along $\overrightarrow{OX'}$ and $\overrightarrow{OY'}$ are considered as negative.

9.1.5 Locating an Ordered Pair *(a,b)*, Recognizing Abscissa and Ordinate

Let XOX' and YOY' be the co-ordinate axes with P, a point in the plane. From P we draw perpendicular to $X'OX$ such that $\overline{OM} = a$ and $\overline{MP} = b$.

Then we say that co-ordinates of P are (a,b). We denote this point by $P(a,b)$. The number "a" is called the x- co-ordinate or abscissa of the point P while b is called the y-co-ordinate or ordinate of the point P.



9.1.6 Geometrical Shapes by Joining a Set of Given Points

EXAMPLES

Draw a line segment, a triangle and a rectangle with the help of the given points: (i) A(−3,−4), B(4,5), (ii) A(2,3), B(−3,4), C(4,−5) (iii) A(4,3), B(−4,3), C(−4, −3), D(4,−3)

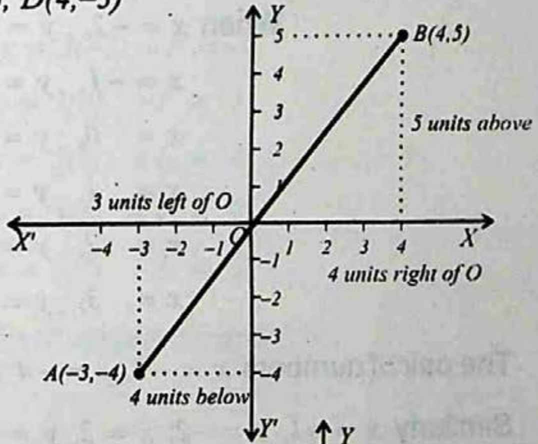
SOLUTION:

(i) The two ordered pairs are:
A(−3,−4) and *B(4,5).*

For point *A(−3,−4):*
We move *3* units towards the left of '*O*' along *X*-axis and then *4* units below *X*-axis.

For point *B(4,5):*
We move *4* units towards right of '*O*' along *X*-axis and then *5* units above *X*-axis.

Join *A* to *B* to obtain line-segment *AB.*



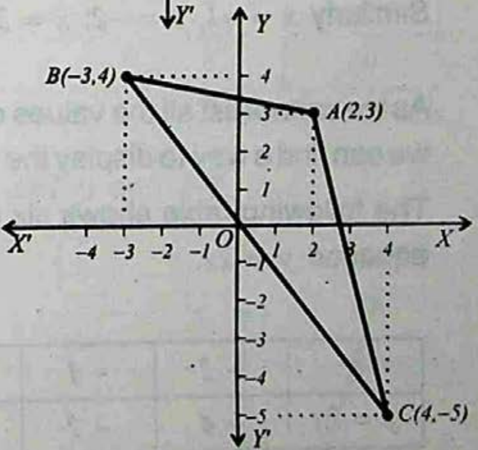
(ii) The given ordered pairs are *A(2,3), B(−3,4)* and *C(4,−5).*

For point *A(2,3):*
We move *2* units towards the right of '*O*' along *X*-axis and *3* units above *X*-axis.

For point *B(−3,4):*
We move *3* units towards the left of '*O*' along *X*-axis and *4* units above *X*-axis.

For point *B(4,−5):*
We move *4* units towards the right of '*O*' along *X*-axis and *5* units below *X*-axis.

We join *A* to *B* : *B* to *C* and *C* to *A* to obtain a triangle *ABC.*



(iii) The given four ordered pairs are:
A(4,3), B(−4,3), C(−4,−3) and *D(4,−3).*

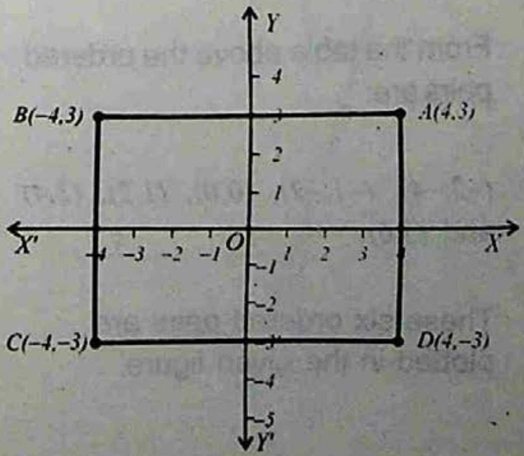
For point *A(4,3):*
We move *4* units towards the right of '*O*' along *X*-axis and then *3* units above *X*-axis.

For point *B(−4,3):*
We move *4* units towards left of '*O*' along *X*-axis and *3* units above *X*-axis.

For point *C(−4,−3):*
We move *4* units towards left of '*O*' along *X*-axis and *3* units below *X*-axis.

For point *D(4,−3):*
We move *4* units towards the right of '*O*' along *X*-axis and *3* units below *X*-axis.

We join *A* to *B* : *B* to *C* : *C* to *D* and *D* to *A* to obtain a rectangle *ABCD.*



9.1.7 Table for Pairs of Values Satisfying a Linear Equation in two Variables

Let us consider the equation $y = 2x$. We look all pairs of numbers x and y that satisfy the equation $y = 2x$.

When $x = -2, \; y = 2(-2) = -4$

$x = -1, \; y = 2(-1) = -2$

$x = 0, \; y = 2(0) = 0$

$x = 1, \; y = 2(1) = 2$

$x = 2, \; y = 2(2) = 4$

$x = 3, \; y = 2(3) = 6$

The pair of numbers $x = -2, y = -4$ satisfies $y = 2x$.

Similarly $x = -1, y = -2; x = 2, y = 4$ etc, satisfy the equation $y = 2x$.

As we cannot list all the values of x and y that satisfy the equation $y = 2x$, we can find a way to display the function $y = 2x$.

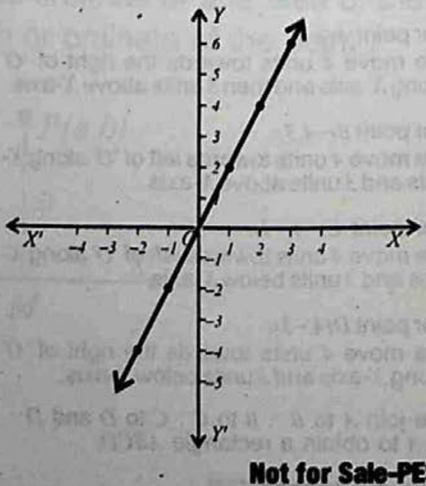
The following table shows six pairs of values of x and y that satisfy the equation $y = 2x$.

x	-2	-1	0	1	2	3
$y = 2x$	-4	-2	0	2	4	6

From the table above the ordered pairs are:

$(-2,-4)$, $(-1,-2)$, $(0,0)$, $(1,2)$, $(2,4)$ and $(3,6)$.

These six ordered pairs are plotted in the given figure.



9.1.8 Plot the Pairs of Points to Obtain the Graph of a given Expression

Let us consider an equation $y = 3x + 1$.
We look at some pair of numbers x and y that satisfy the equation $y = 3x + 1$.

When $x = -1 \Rightarrow y = 3(-1) + 1 = -2$

$x = 0 \Rightarrow y = 3(0) + 1 = 1$

$x = 1 \Rightarrow y = 3(1) + 1 = 4$

$x = 2 \Rightarrow y = 3(2) + 1 = 7$

$x = 3 \Rightarrow y = 3(3) + 1 = 10$

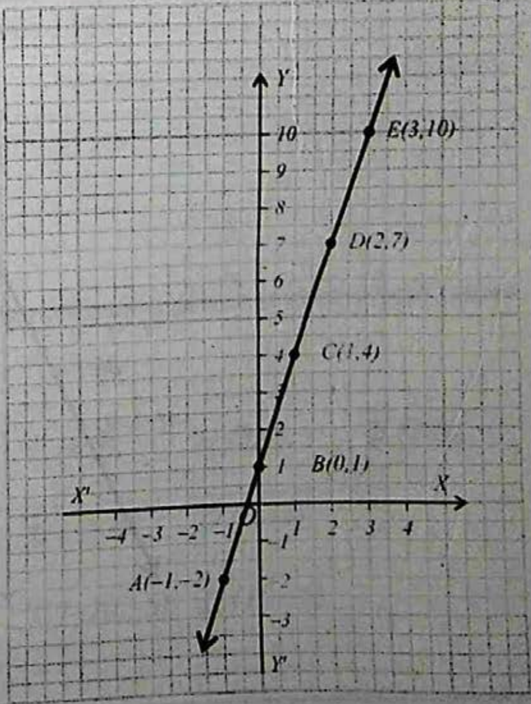
The table below shows five pairs of values of x and y.

x	-1	0	1	2	3
$y = 3x + 1$	-2	1	4	7	10

The five ordered pairs are :

$A(-1,-2)$, $B(0,1)$, $C(1,4)$, $D(2,7)$ and $E(3,10)$

Now we plot these pairs on the graph paper.



For point $A(-1,-2)$:
We move 1 unit towards left of 'O' along X-axis and 2 units below X-axis.

For point $B(0,1)$:
We move 1 unit along Y-axis above 'O'.

For point $C(1,4)$:
We move 1 unit towards right of 'O' along X-axis and 4 units above X-axis.

For point $D(2,7)$:
We move 2 units towards right of 'O' along X-axis and 7 units above X-axis.

For point $E(3,10)$:
We move 3 units towards the right of 'O' along X-axis and 10 units above X-axis.

We draw a line AE.

9.1.9 Choosing an Appropriate Scale to Draw a Graph

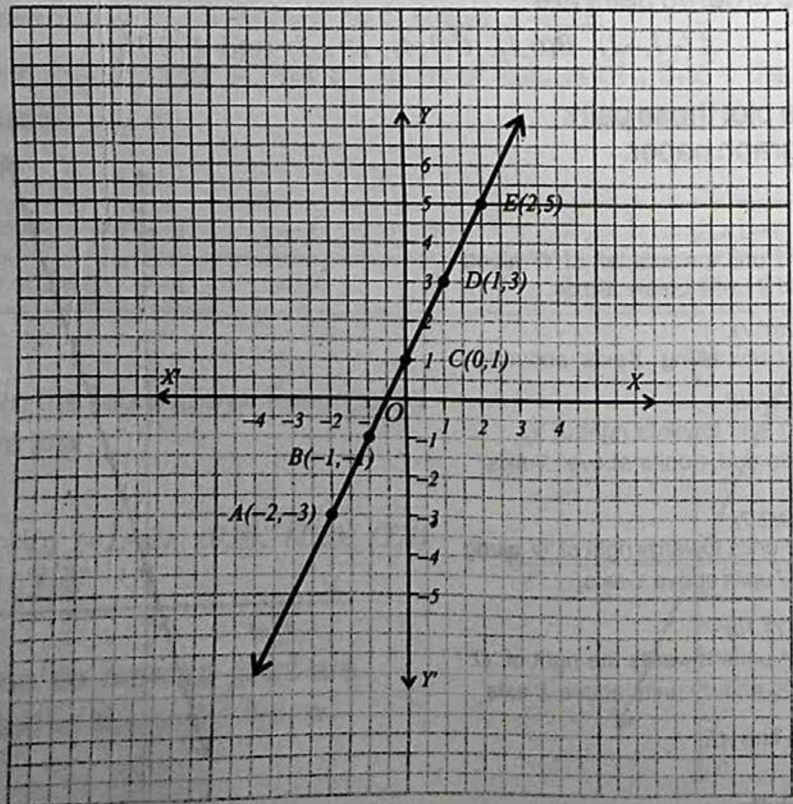
Let us consider the equation $y = 2x + 1$.

$$\text{When } x = -2, \quad y = 2(-2) + 1 = -3$$
$$x = -1, \quad y = 2(-1) + 1 = -1$$
$$x = 0, \quad y = 2(0) + 1 = 1$$
$$x = 1, \quad y = 2(1) + 1 = 3$$
$$x = 2, \quad y = 2(2) + 1 = 5$$

The following table shows five pairs of values of x and y mentioned above.

x	-2	-1	0	1	2
$y = 2x + 1$	-3	-1	1	3	5

We use two small squares (or 1 big square) on the graph paper to represent both x and y.



EXERCISE – 9.1

1- Represent the points on the graph whose co-ordinates are given below.

(i) $A(2,-4)$ (ii) $B(3,2)$

(iii) $C(-5,-1)$ (iv) $D(6,-3)$

(v) $E(4,4)$ (vi) $F(-3,7)$

(vii) $G(0,7)$ (viii) $H(5,0)$

2- Write down the co-ordinates of:

(i) Origin

(ii) A point lying on the left hand side of *x-axis* and at a distance of 5 units from the origin.

(iii) A point lying to the right hand side of the origin on *x-axis* at a distance of 3 units from the origin.

(iv) A point lying above *x-axis* and on *y-axis* at a distance of 4 units.

(v) A point lying below *x-axis* and on *y-axis* at a distance of 6 units.

3- Draw the figures with help of the following points on the graph paper.

(i) $A(7,2)$, $B(-6,-3)$, $C(5,3)$

(ii) $A(0,-7)$, $B(3,-2)$, $C(4,0)$, $D(5,6)$, $E(7,8)$

(iii) $A(4,0)$, $B(0,4)$, $C(-4,0)$, $D(0,-4)$

(iv) $A(10,6)$, $B(-10,6)$, $C(-10,-6)$, $D(10,-6)$,

9.1.10 Graphs of Linear Equations of the form $y = c$

Graph of a Linear Equation of the form $y = c$

To draw the graph of $y = c$, we can write the equation $y = c$ in the form $0 . x + y = c$. Procedure to draw the graph is explained through the following example:

EXAMPLE

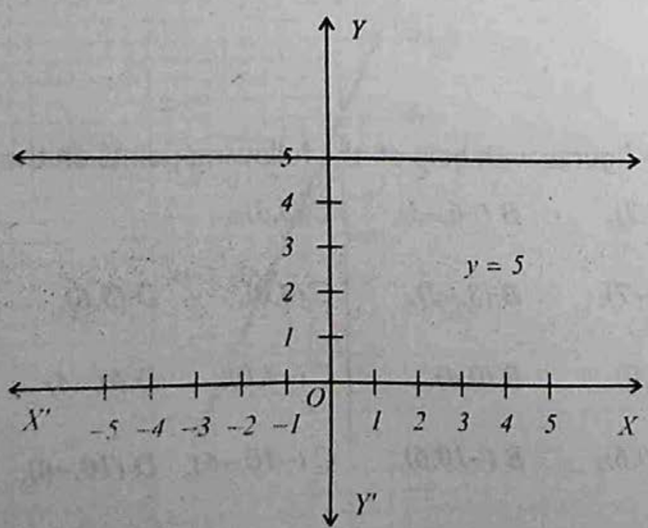
Draw the graph of an equation $y = 5$.

SOLUTION:

The equation $y = 5$ can be written as $y = 0 \times x + 5$
If we put $x = 0$ in the equation we get $y = 5$.
Similarly putting $x = \pm 1, \pm 2, \pm 3, ...$ in the equation $y = 0 . x + 5$, we have $y = 5$. For all values of x we have $y = 5$, i.e. y remains constant.

Table of values of x and y is as under:

x	-3	-2	-1	0	1	2	3
y	5	5	5	5	5	5	5



Graph of a Linear Equation of the form $x = a$

To draw the graph of $x = a$, we can write the equation $x = a$ in the form of $x + 0 \, . \, y = a$ procedure to draw the graph is explained in the following example:

EXAMPLE

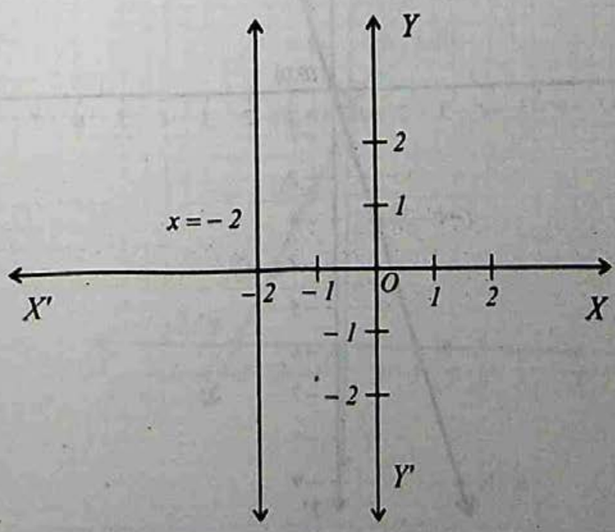
Draw the graph of an equation $x = -2$.

SOLUTION:

The equation $x = -2$ can be written as $x + 0 \, . \, y = -2$. If we put $y = 0$ in this equation, we get $x = -2$. Similarly putting $y = \pm 1, \pm 2, \pm 3, ...$ in the equation $x + 0 \, . \, y = -2$, we have $x = -2$.
For all values of y we have $x = -2$, i.e. x remains constant.

Table of values of x and y is as under:

x	−2	−2	−2	−2	−2	−2	−2
y	−3	−2	−1	0	1	2	3



Graph of a Linear Equation $y = mx$

To draw the graph of $y = mx$, we consider the following example:

EXAMPLE

Draw the graph of $y = 3x$.

SOLUTION:

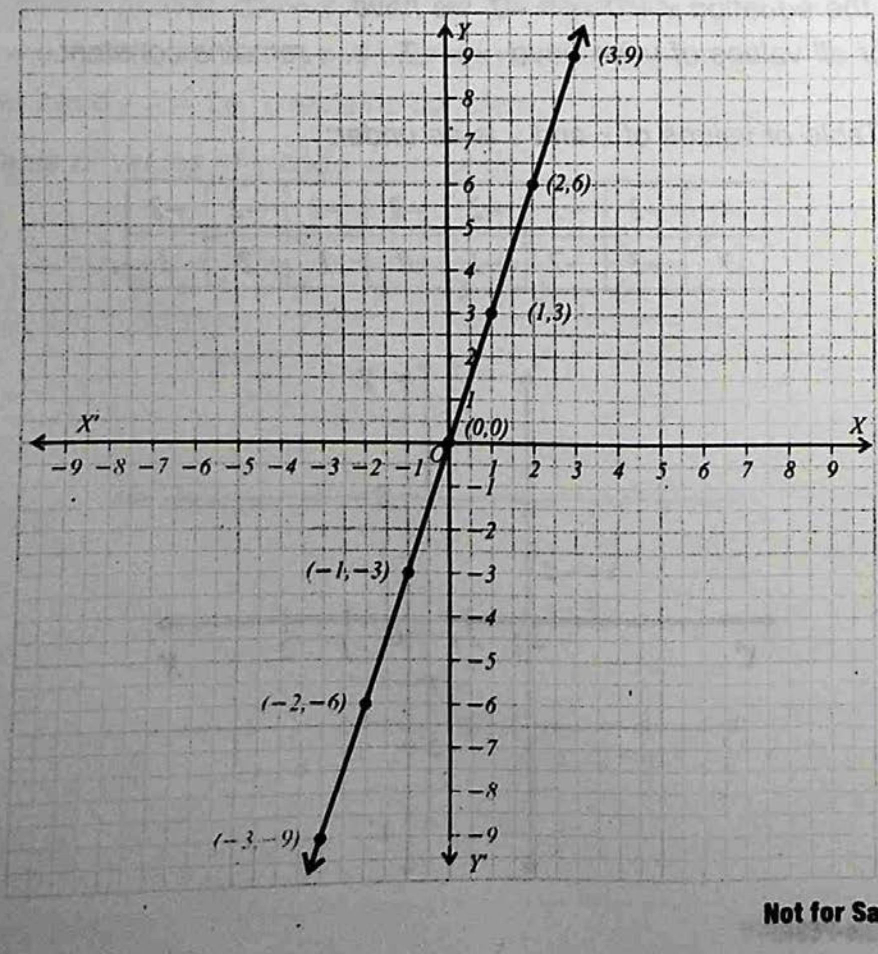
To draw the graph of $y = 3x$, first we find the x and y–intercept of this equation.

Putting $x = 0$ in $y = 3x$, we get $y = 0$. The point of interception is $(0,0)$.

Putting $x = \pm 1, \pm 2, \pm 3, \ldots$ in $y = 3x$, we get $y = \pm 3,\ y = \pm 6,\ y = \pm 9, \ldots$

Table of values of x and y is as under:

x	-3	-2	-1	0	1	2	3
y	-9	-6	-3	0	3	6	9



Graph of an Equation $y = mx + c$

To draw the graph of an equation $y = mx + c$, we consider the following example:

EXAMPLE

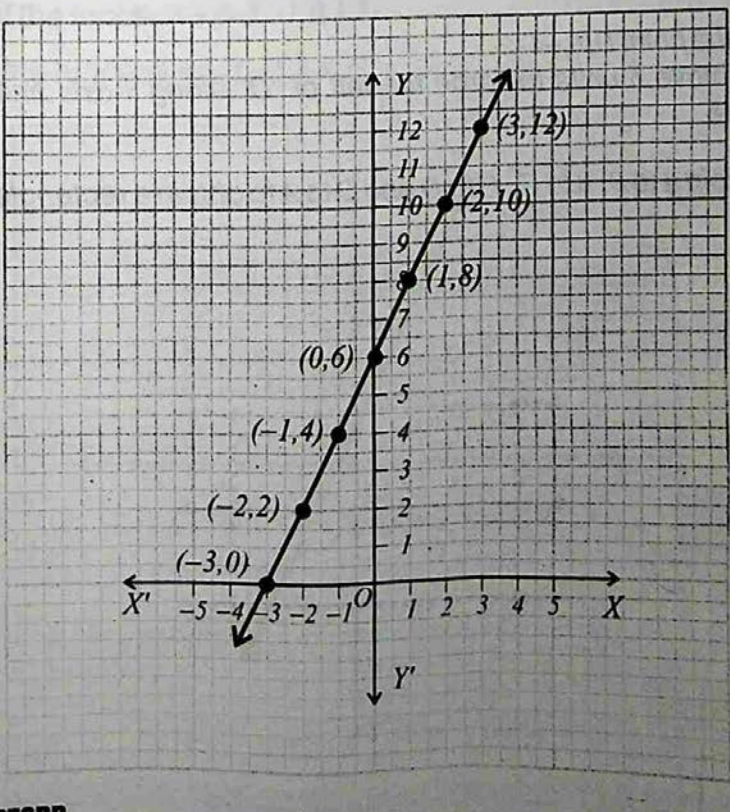
Draw the graph of $y = 2x + 6$.

SOLUTION:

If we put $x = 0$ in $y = 6 - 2x$, we get $y = 2 \times 0 + 6 = 6$ i.e. $y = 6$.

Similarly putting $x = \pm 1, \pm 2, \pm 3, ...$ we get the values of y as shown in the table.

x	-3	-2	-1	0	1	2	3
y	0	2	4	6	8	10	12



9.1.11 Draw a Graph from a given Table

We plot the following points on a piece of graph paper.

x	6	−6	−6	6
y	4	4	−4	−4

The four ordered pairs from the table are:
A(6,4), *B(−6,4)*, *C(−6,−4)*, and *D(6,−4)*.

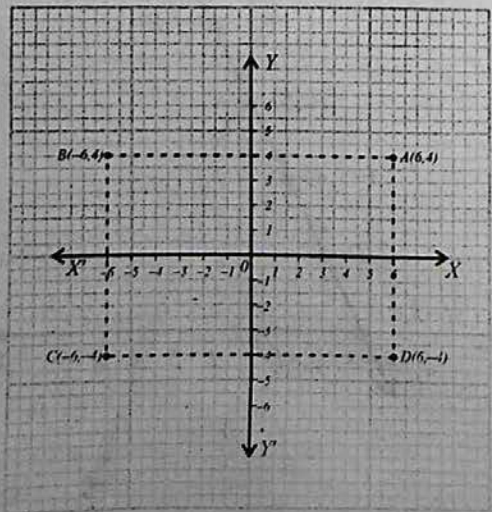
For the point A(6,4):
We move 6 units towards the right of 'O' along x-axis and 4 units below x-axis.

For the point B(−6,4):
We move 6 units towards the right of 'O' along x-axis and 4 units above x-axis.

For the point C(−6,−4):
We move 6 units towards the left of 'O' along x-axis and 4 units above x-axis.

For the point D(6,−4):
We move 6 units towards the left of 'O' along x-axis and 4 units below x-axis.

We join A to B; B to C; C to D and D to A to obtain a rectangle ABCD.



9.1.12 Identification of Domain and Range of a Function Through Graph

The graph shown in the figure is of a function $y = 2x + 1$. This graph has been drawn with the help of the following ordered pairs. $A(-2,-3)$, $B(-1,-1)$, $C(0,1)$, $D(1,3)$ and $E(2,5)$.

From these ordered pairs we construct a table consisting the values of x and y.

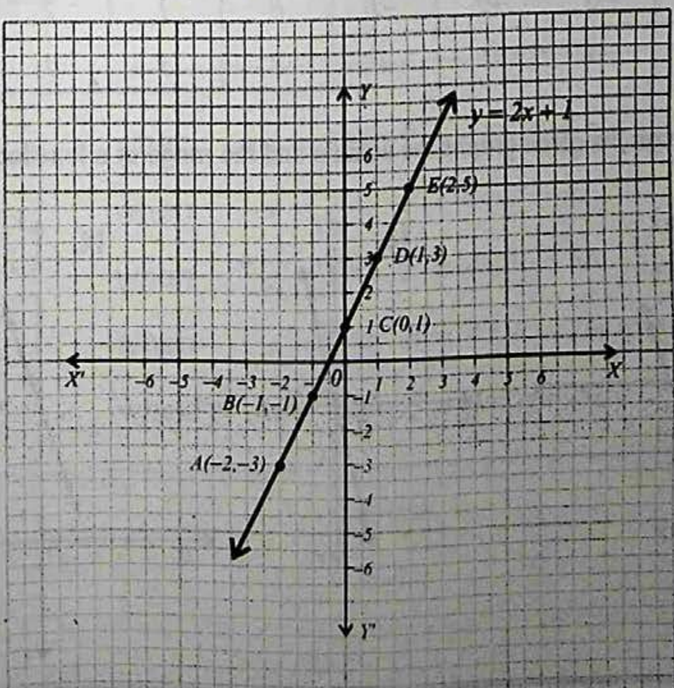
x	-2	-1	0	1	2
y	-3	-1	1	3	5

In a function $y = 2x + 1$, the set consisting of the values of x is called the domain and the set consisting the values of y is called the range of the function.

Thus for $y = 2x + 1$:

Domain of the function $= \{-2,-1,0,1,2\}$

Range of the function $= \{-3,-1,1,3,5\}$



EXERCISE – 9.2

Draw the graph of:

1. $y = 3x$
2. $y = x + 7$
3. $y = 2x - 3$
4. $y = 4x + 1$
5. $y = -\frac{x}{2} - \frac{3}{2}$
6. $y = x - 1$
7. $y = 2x - 3$
8. $y = 3x + 5$
9. $y = \frac{x}{2}$

10. Draw the graph by plotting the points $A(1,0)$, $B(7,0)$ and $C(1,8)$.

11. Draw the graph from the given tables.

(i)

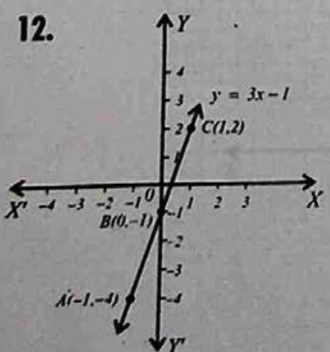
x	3	2	1	0	−1	−2	−3
y	−5	−3	−1	1	3	5	7

(ii)

x	−3	−2	−1	0	1	2	3
y	−1	0	1	2	3	4	5

Identify through the given graphs the domain and the range of a function

12.



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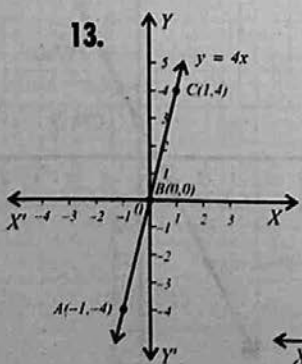

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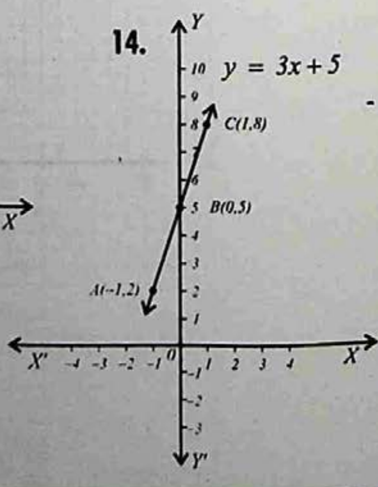


9.2 CONVERSION GRAPHS

If we have to go to London for holidays, we would probably have a little difficulty in knowing the cost of things in pounds and pence. If we know the rate of exchange, we can use a simple straight line graph to convert a given number of rupees into pounds or a given number of pounds into rupees.

The straight line used for this purpose is called the conversion graph.

9.2.1 Conversion Graph as a Linear Graph

The perimeter of a square is given by the formula $P = 4s$, where "P" units are the perimeter and "s" units are the length of the sides. This is an example of direct proportion, because by changing one quantity, the other also changes.

For example:

when $s = 1 \Rightarrow P = 4 \times 1 = 4,$

when $s = 2 \Rightarrow P = 4 \times 2 = 8,$

when $s = 3 \Rightarrow P = 4 \times 3 = 12,$

when $s = 4 \Rightarrow P = 4 \times 4 = 16,$

when $s = 5 \Rightarrow P = 4 \times 5 = 20,$

9.2.2 Read a given Graph to Know One Quantity Corresponding to Another

Let us consider the following examples:

EXAMPLE-1

The graph shows the conversion from US dollars ($) to pounds (£) for various amounts of money.

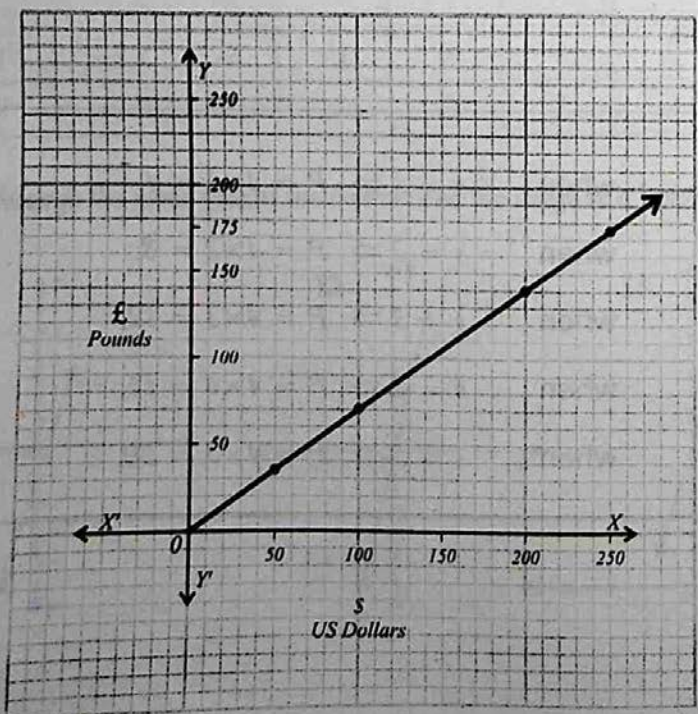
$	50	100	200	250
£	35	70	140	175

SOLUTION: (i) 50 dollars are converted into 35 pounds.

(ii) 100 dollars are converted into 70 pounds.

(iii) 150 dollars into 105 pounds.

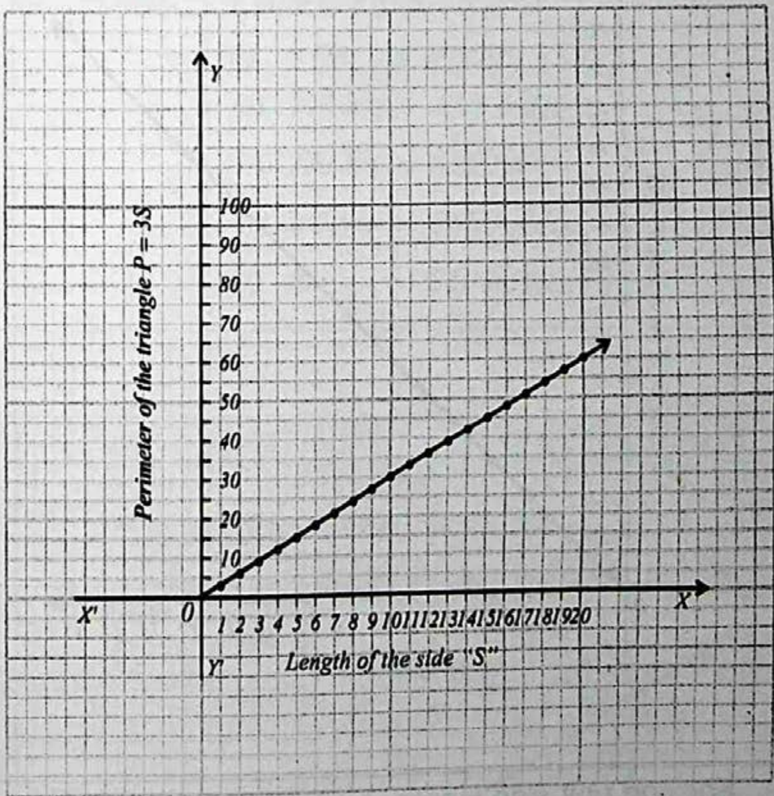
(iv) 250 $ into 175 pounds.



EXAMPLE-2

The graph is the relationship between the side length and the perimeter of an equilateral triangle $P = 3S$ for values of 'S' from 1 to 20.

S	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20
$P = 3S$	3	6	9	12	15	18	21	24	27	30	33	36	39	42	45	48	51	54	57	60

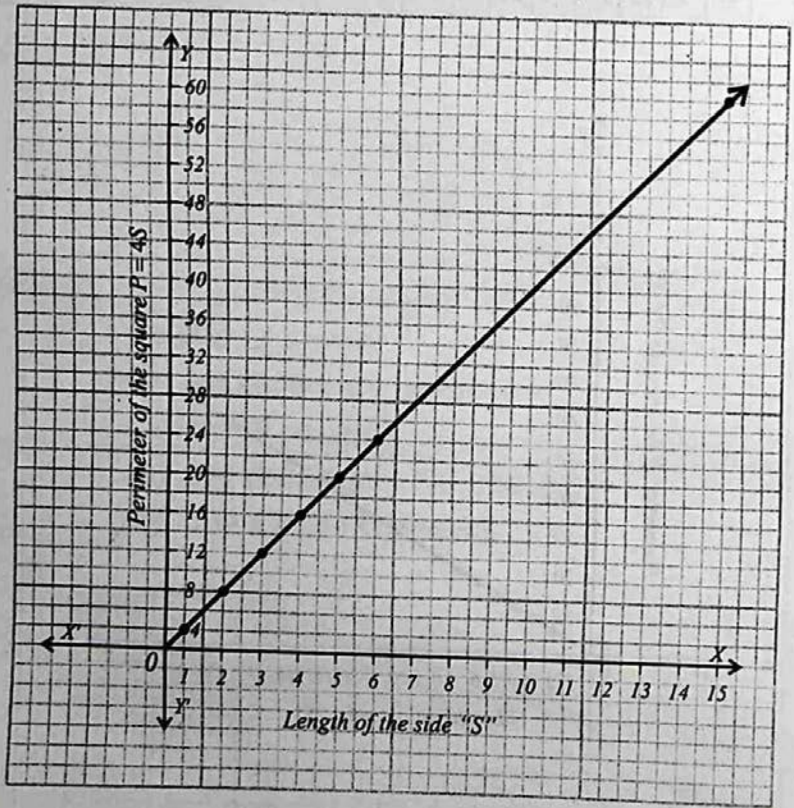


(i) For one unit side length, perimeter is 3 units.
(ii) For 3 units side length, perimeter is 9 units.
(iii) For $S = 4$, $P = 12$
(iv) For $S = 6$, $P = 18$
(v) For $S = 9$, $P = 27$
(vi) For $S = 11$, $P = 33$
(vii) For $S = 14$, $P = 42$
(viii) For $S = 16$, $P = 48$
(ix) For $S = 20$, $P = 60$

EXAMPLE-3

The graph is the relationship between the side length and the perimeter of a square $P = 4S$ for values of 'S' from 1 to 15.

S	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
$P = 4S$	4	8	9	12	16	20	24	28	32	36	40	44	48	52	60



(i) For one unit side length, perimeter is 4 units.

(ii) For 4 units side length, perimeter is 8 units.

(iii) For $S = 3$, $P = 12$

(iv) For $S = 4$, $P = 16$

(v) For $S = 5$, $P = 20$

(vi) For $S = 6$, $P = 24$

(vii) For $S = 7$, $P = 28$

(viii) For $S = 8$, $P = 32$

(ix) For $S = 9$, $P = 36$

(x) For $S = 10$, $P = 40$

(xi) For $S = 11$, $P = 44$

(xii) For $S = 12$, $P = 48$

(xiii) For $S = 13$, $P = 52$

(xiv) For $S = 14$, $P = 56$

(xv) For $S = 15$, $P = 60$

9.2.3 Read the Graph for Conversion

Miles and Kilometers

Read the conversion graph.

Conversion: *1 mile = 1.6 km*

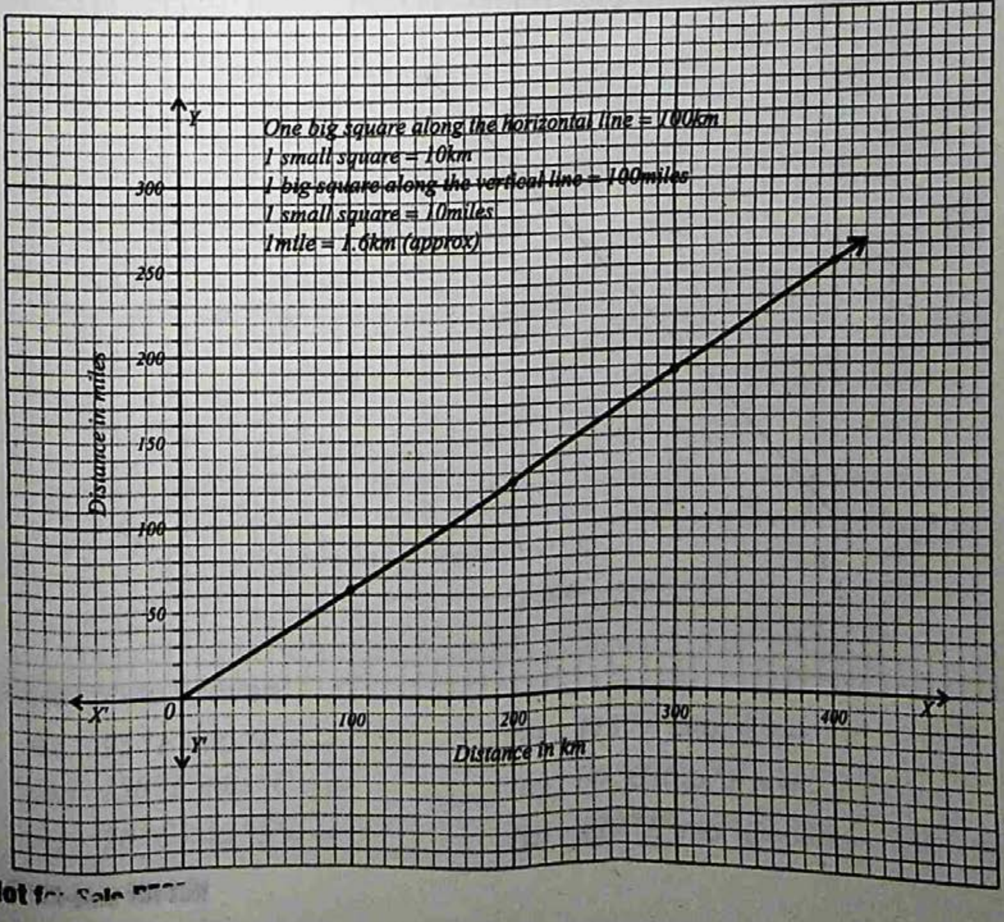
(i) 0 km = 0 miles

(ii) 100 km = 62.5 miles

(iii) 200 km = 125 miles

(iv) 300 km = 187.5 miles

(v) 400 km = 250 miles

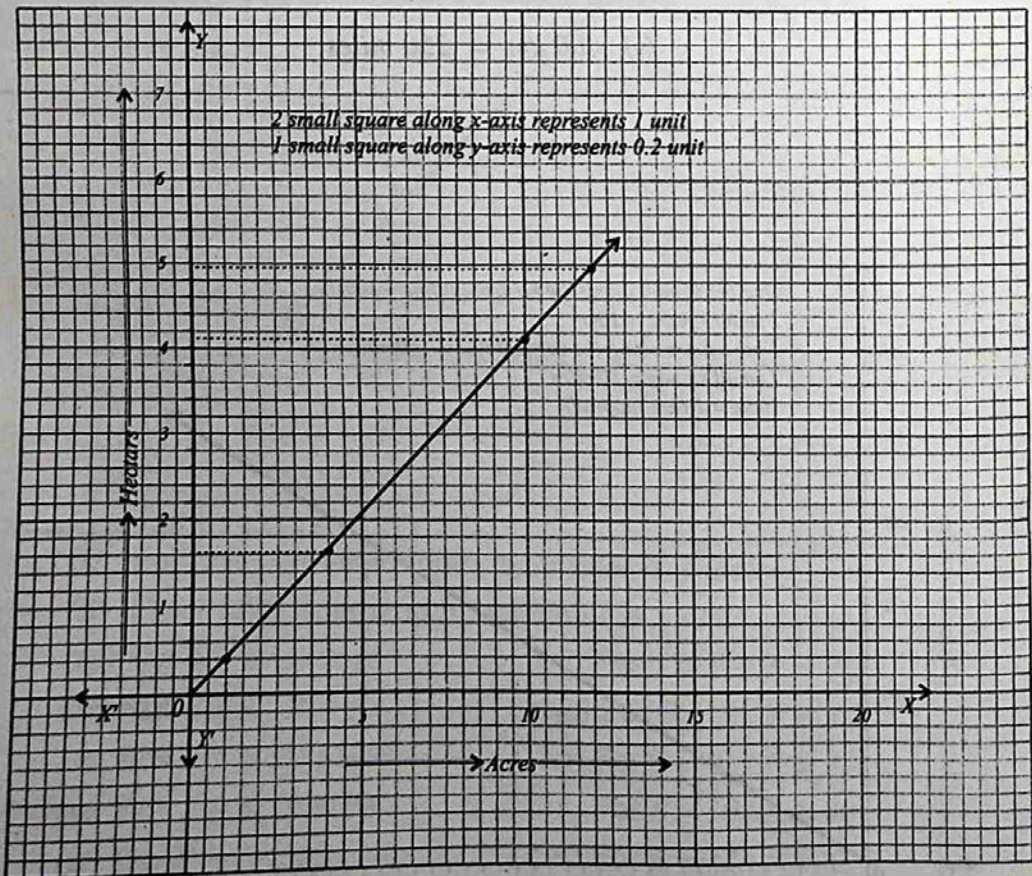


Acres and Hacters

The table gives area in acres and the equivalent values in hectares.

Acres	*1*	*4*	*10*	*12*
Hectares	*0.4046*	*1.6187*	*4.0468*	*4.8562*

Plot these points on the graph for acres values from *0* to *30* and hectares values from *0* to *12.1405*. Let *2* small squares represents one unit along *x-axis*. One small square along *y-axis* represents *0.2* units.

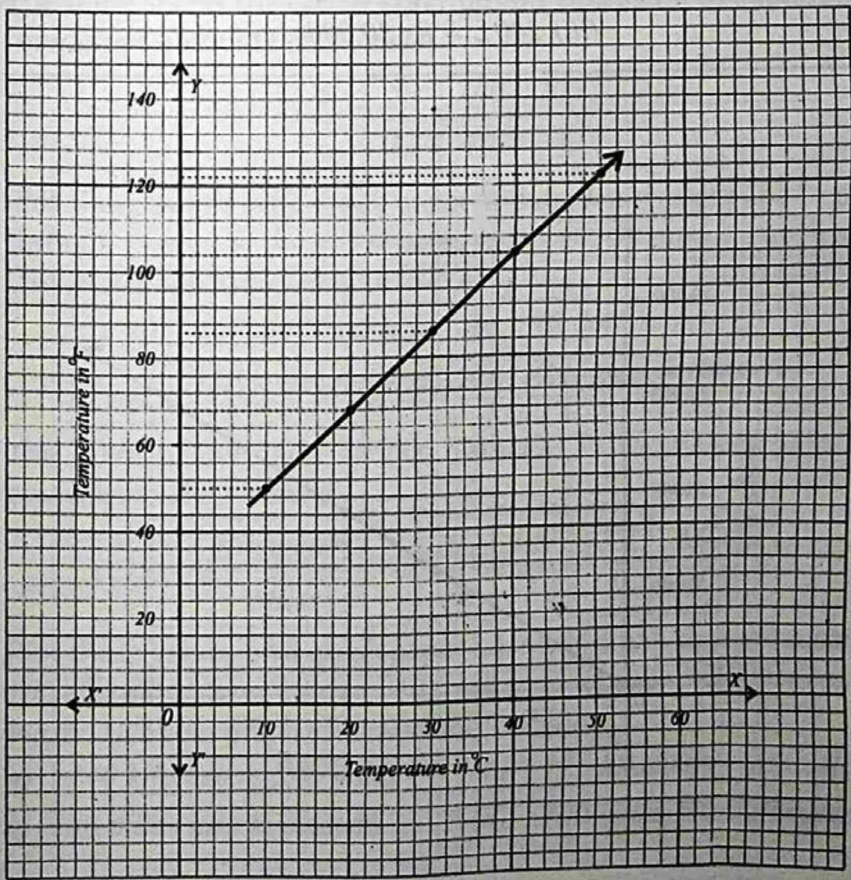


Degrees Celsius and Degrees Fahrenheit

The graph gives temperature in degrees Fahrenheit *(F)* and the equivalent values in degrees Celsius ℃. Read the graph carefully and answer the questions.

Temperature in degrees Celsius from *0* to *50* is along the horizontal line, where as temperature in degrees Fahrenheit is along the vertical line. *5* small square along *x-axis* represents *10* ℃ and *5* small squares along *y-axis* represents *20*°F.

Conversion: $^{o}C = \frac{5}{9}(^{o}F - 32)$, $^{o}F = \left(\frac{9}{5} \times ^{o}C\right) + 32$



Use graph to convert

(i) $95\,^{o}F$ into $^{o}C = \cdots$ (ii) $113\,^{o}F$ into $^{o}C = \cdots$

(iii) $150\,^{o}F$ into $^{o}C = \cdots$ (iv) $86\,^{o}C$ into $^{o}F = \cdots$

(v) $20\,^{o}C$ into $^{o}F = \cdots$ (vi) $220\,^{o}F$ into $^{o}C = \cdots$

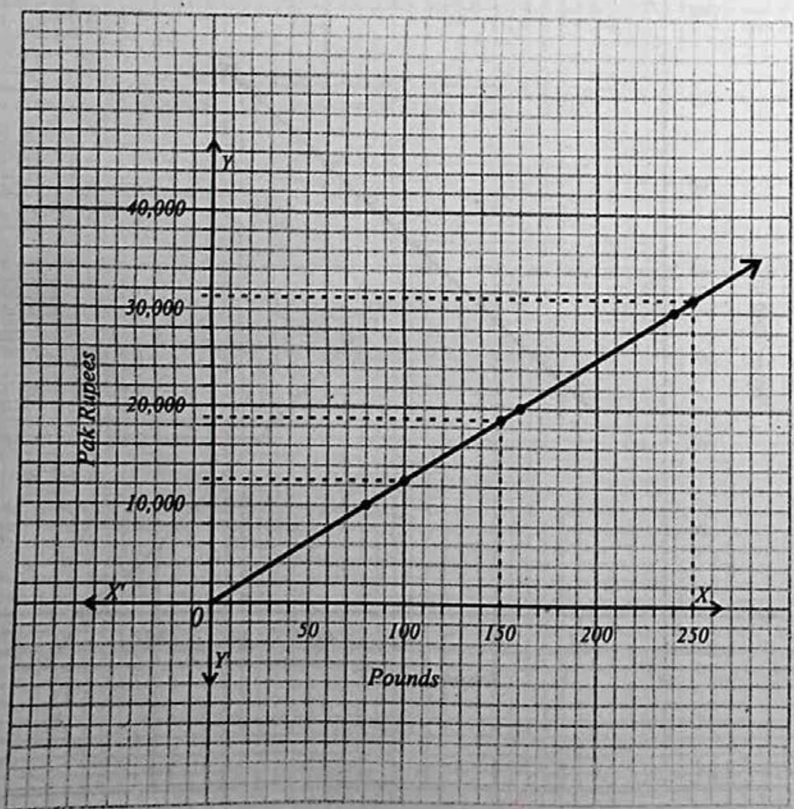
Pakistani Currency and another Currency

The graph shows the conversion from *UK* pounds £ to Pakistani rupees for various amounts of money.

Let *5* small squares on horizontal line represent *50£* and *5* small squares on vertical line represent *10,000* rupees. *1* pound = *Rs.125.*

From that graph:

(i) 80£ = Rs 10,000
(ii) 160£ = Rs 20,000
(iii) 240£ = Rs 30,000



Read the graph and tell ?

*(iv)250£ = Rs.*____ *(v) 150£ = Rs.*____ *(vi) 100£ = Rs.*____

*(vii) Rs 5000 = £*____ *(viii) Rs 8000 = £*____

EXERCISE – 9.3

1. The table gives temperatures in degrees Fahrenheit ^{o}F and the equivalent values in degrees Centigrade ^{o}C.

Temperature in ^{o}F	57	126	158	194
Temperature in ^{o}C	14	52	70	90

Plot these points on a graph paper for centigrade values from *0* to *100* and Fahrenheit value from *0* to *220*. Let *5* small squares represent *20* units on each axis. Use your graph to convert the following:

a) $97\,^{o}F$ *into* ^{o}C, b) $127\,^{o}F$ *into* ^{o}C, c) $25\,^{o}C$ *into* ^{o}F, d) $80\,^{o}C$ *into* ^{o}F

2. The table shows the conversion from US Dollars ($) to Pounds (£) for various amounts of money.

$	50	100	200
£	35	70	140

Plot these points on a graph paper and draw a straight line to pass through them. Let *5* small squares represent *50* units on each axis. Use your graph to convert the following:

a) *160 dollars into £* b) *96 dollars into £*
c) *120 £ into dollars* d) *76£ into dollars*

3. The table below gives various distances in kilometers with the equivalent values in miles.

Kilometers	0	100	200	300
Miles	0	62.5	125	187.5

Plot these values on a graph paper taking *10* small squares equal to *100* kilometers on *x-axis* and *10* small squares equal to *100* miles on *y-axis*. Use your graph to convert the following:

a) *140 kilometers into miles* b) *175 kilometers into miles*
c) *50 miles into kilometers* d) *100 miles into kilometers*

4. Use the graph in article *9.2.3* to convert:
(a) *6 acres into hectares.* (b) *18 acres into hectares.*
(c) *24 acres into hectares.* (d) *6.0702 hectares into acres.*
(e) *11.3311 hectares into acres.*

Review Exercis – 9

1- Encircle the correct answer.

(i) The co-ordinates of origin are:

(a) $(1,1)$ (b) $(0,1)$ (c) $(0,0)$ (d) $(1,0)$

(ii) The perpendicular distance of a point from y-axis is called

(a) *ordinate* (b) *abscissa* (c) *origin* (d) *straight line*

(iii) The perpendicular distance of point from x-axis is called

(a) *ordinate* (b) *abscissa* (c) *origin* (d) *straight line*

(iv) For $x = 1$ in $2x + y = 6$, we have $y = ?$

(a) 8 (b) 4 (c) -8 (d) -4

(v) For $y = 2$ in $2x - y = 6$, we have $x = ?$

(a) 4 (b) -4 (c) 2 (d) -2

(vi) Graph of equation in the form $y = c$ has y co-ordinate:

(a) 1 (b) c (c) 0 (d) -1

(vii) Graph of equation in the form $x = a$ has x co-ordinate:

(a) a (b) *undefined* (c) 1 (d) c

(viii) $f(x) = \frac{x}{2}, 4 \leq x \leq 12,$ where x is a multiple of "2".
The domain of $f(x)$ is:

(a) $\{4,6,8,10,12\}$ (b) $\{6,8,10\}$ (c) $\{4,6,8,10\}$ (d) $\{2,3,4,5,6\}$

(ix) $f(x) = \frac{x}{2}, 4 \leq x \leq 12,$ where x is a multiple of "2".
The range of $f(x)$ is:

(a) $\{4,6,8,10,12\}$ (b) $\{2,3,4,5,6\}$ (c) $\{3,4,5\}$ (d) $\{2,3,4,5,6\}$

(x) If $y = 3x$, then for $x = 2$, we have $y = ?$:

(a) 0 (b) 6 (c) -3 (d) 2

2- Fill in the blanks.

(i) A plane consisting of two number lines OX and OY intersecting at right angle at "O" is called a __________.

(ii) The perpendicular distance of a point from y-axis is called __________.

(iii) The perpendicular distance of a point from x-axis is called __________.

(iv) The pair of number $(2,3)$ is called an __________.

(v) The horizontal line $X'OX$ is called __________.

(vi) The vertical line YOY' is called __________.

(vii) For a point $(-1,-2)$ we move 1 unit towards left of "O" and 2 units __________.

(viii) The co-ordinate of origin are__________.

(ix) An equation for a straight line that consist of y term is as__________.

(x) In the graph of $2x + y = 6$, the x-intercept is __________

3- Draw the figures with the help of the following points on the graph paper.

(i) $A(5,2)$, $B(-6,-3)$ and $C(3,5)$

(ii) $A(0,-5)$, $B(3,-2)$, $C(3,0)$ and $D(6,7)$

(iii) $A(8,4)$, $B(-6,3)$, $C(-5,-3)$ and $D(10,-6)$.

4- Sketch the graph of:

(i) $y = 3x + 2$

(ii) $y = 2x + 1$

(iii) $y = x + 1$

(iv) $y = -\frac{x}{2} - \frac{5}{2}$

(v) $y = 3x + 4$

5- Draw the graph by plotting the points $A(2,0)$, $B(7,0)$ and $C(1,8)$.

SUMMARY

- By the definition of equality of sets, for any two elements *"a"* and *"b"*, we have $\{a,b\} = \{b,a\}$.

- The pairs of numbers *(2,1), (2,2), (2,3), (2,4), (3,1), (3,3)* and so on are examples of ordered pairs.

- We can use a simple straight line graph to convert a given number of rupees into pounds or a given number of pounds into rupees.

UNIT 10

BASIC STATISTICS

- **Frequency Distribution**
- **Cumulative Frequency**
- **Measure of Central Tendency**
- **Measure of Dispersions**

After completion of this unit, the students will be able to:

- Construct grouped frequency table.
- Construct histograms with equal and unequal class intervals.
- Construct a frequency polygon.
- Construct a cumulative frequency table.
- Draw a cumulative frequency polygon.
- Calculate (for ungrouped and grouped data):
  - Arithmetic mean by definition and using deviations from assumed mean,
  - Median, mode, geometric mean, harmonic mean.
- Recognize properties of arithmetic mean.
- Calculate weighted mean and moving averages.
- Estimate median, quartiles and mode graphically.
- Measure range, variance and standard deviation.

10.1 FREQUENCY DISTRIBUTION

Frequency

The number of times each value appears is called the frequency of that value. For example if *3* students get marks from *10%* to *20%*, the frequency is *3* if *5* students get marks from *20%* to *30%*, frequency is *5*.

Frequency Table

The table which gives the frequency of each score is called a frequency table.

10.1.1 Grouped Frequency Table

The local health authority wanted to collect some information about the heights of five years old children in their area. At their first visit to school for medical examination, the height of each child of five years was recorded, e.g., the following figures are the recorded heights *(in cm to the nearest cm)* of ninety children.

99	107	102	98	115	95	106	110	108	105
118	102	114	108	94	104	113	102	105	95
105	110	109	101	106	108	107	107	101	109
108	105	116	109	114	110	97	110	113	116
112	101	92	105	104	115	111	103	110	99
93	104	103	113	107	94	102	117	116	104
99	114	106	114	98	109	107	104	106	107
109	113	112	100	109	113	118	104	94	114
107	96	108	103	112	106	115	111	115	101

In the table the heights are listed in a random order *(the same order the children came for the medical examination)*. We are to arrange and organize this data in some manner for clear understanding to get any result. We group the heights as:

90 – 94cm ; *95 – 99cm* ; *100 – 104cm* ;
105 – 109cm ; *110 – 114cm* ; *115 – 119cm*

The heights are arranged in order of magnitude. Counting the number of heights is each group gives the following frequency table.

Heights *(cm)*	Tally	Frequency
90 – 94	𝍸	5
95 – 99	𝍸 ||||	9
100 – 104	𝍸 𝍸 𝍸 ||	17
105 – 109	𝍸 𝍸 𝍸 𝍸 𝍸 |||	28
110 – 114	𝍸 𝍸 𝍸 𝍸 |	21
115 – 119	𝍸 𝍸	10
		Total: 90

10.1.2 Histogram

When a bar chart is constructed so that the area of each bar is proportional to the number of items in each group, it is called a histogram.

Histogram with equal class intervals

Consider the following examples:

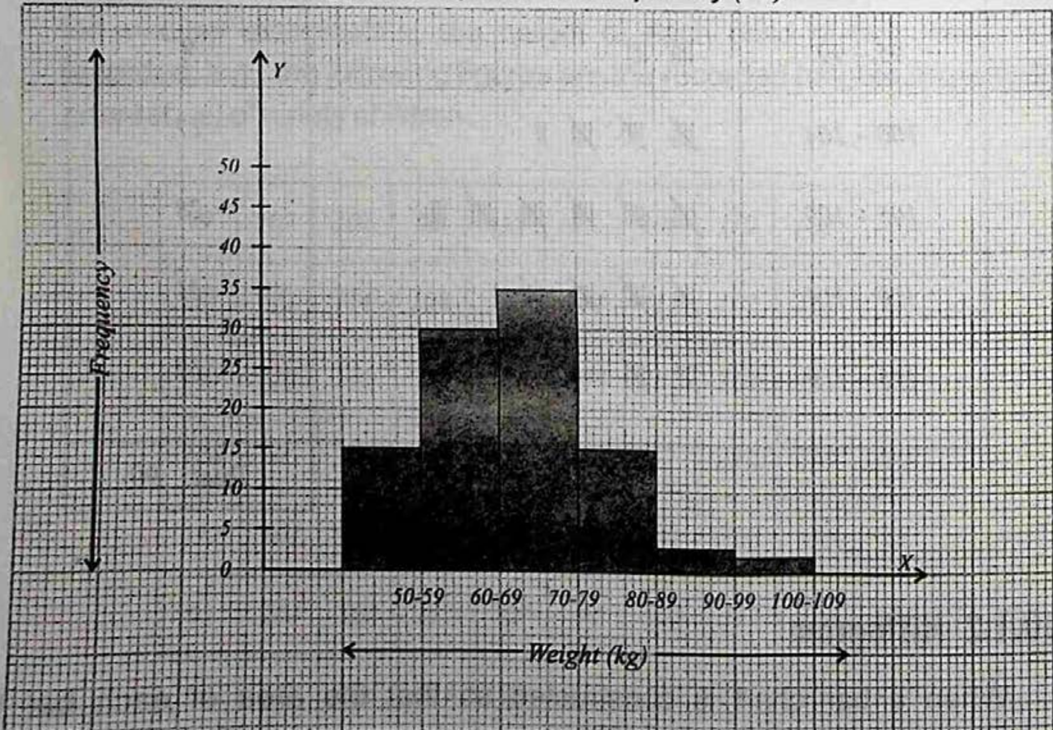
EXAMPLE

Construct a histogram with equal intervals with the help of following frequency.

Weights (kg)	50 – 59	60 – 69	70 – 79	80 – 89	90 – 99	100 – 109
Frequency	15	30	35	15	3	2
					Total:	100

SOLUTION:

The following bar chart illustrates the frequency table.
Weight (kg) has been taken along *x-axis* towards right of 'O'.
Frequency has been taken above *x-axis*.
One big square along *x-axis* represents 10kg weight.
One big square along *y-axis* represents frequency (10).



It is the area of a bar that gives the impression of the number of items in a group.

The diagram is a histogram with equal intervals. Each group covers the same span, so each bar has the same width. Hence the area of each bar is proportional to the number of items in the group.

Histogram with unequal class intervals

EXAMPLE

Consider the following frequency table which shows the result of a survey on the per month pay of 100 men.

Pay (rupees)	0 – 1999	2000 – 3999	4000 – 5999	6000 – 9999	10000 – 19999
Frequency	20	36	25	14	5

SOLUTION:

The intervals of the group are not equal.
The first three groups each cover a span of Rs.2000.
The fourth group intervals Rs.40000 twice the interval of the first three.
The fifth group intervals Rs.10,000 five times the interval of the first three.

To illustrate this distribution on histogram we must make:

The width of the first three bars equal.

The width of the fourth bar twice that of the first three.

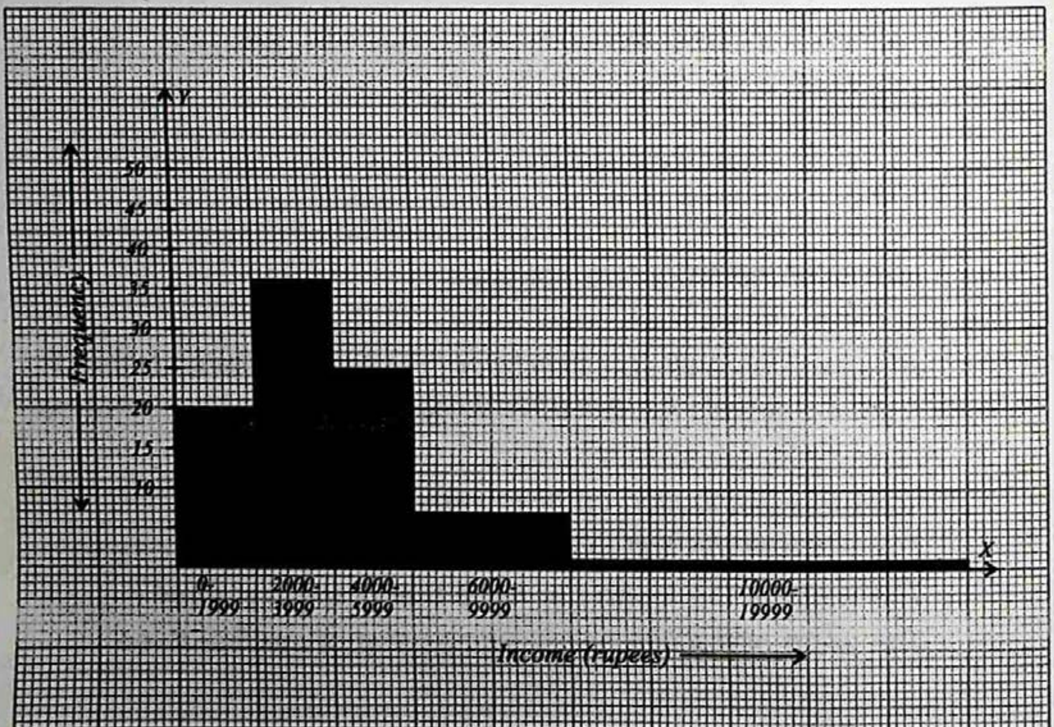
The width of the fifth bar five times that of the first three.

Now consider the 14 incomes in the group Rs.6000 - Rs.9999. If we suppose that these incomes are evenly spread throughout this group, then these are:

Seven men with incomes in the group Rs.6000 - Rs.7999
and seven men with incomes in the group Rs.8000 - Rs.9999.

The fourth bar is therefore 7 units high for the whole group, because the width of this bar is twice that of the others used. Its area is proportional to the frequency of the group.

If we also assume that the income in the fifth group are evenly spread throughout the group, then there is one man with an income in each of the subgroups with interval Rs.2000. The fifth bar is therefore one unit high for the whole group. The area of the fifth bar is proportional to the frequency of the group.



Note: *We do not use frequency to label vertical axis, because it is the area of the bar which represents the frequency and not the height.*

10.1.3 Frequency Polygon

A frequency polygon is a many sided closed figure, It is constructed by plotting frequencies against their class marks (midpoints) and then joining the resulting points by means of straight lines. A frequency polygon can also be obtained by joining the mid points of the tops of the rectangles in the histogram.

EXAMPLE

The data in the frequency table shows the mass (in kg) of 40 people upon joining a weight loss program.

Represent the given data using a:

(i) Histogram (ii) Frequency Polygon

Class intervals	Frequency
60 —— 70	2
70 —— 80	5
80 —— 90	9
90 —— 100	12
100 —— 110	7
110 —— 120	3
120 —— 130	2
	Total: 40

SOLUTION:

(i) Draw the axes on graph paper.

(ii) Title the graph.

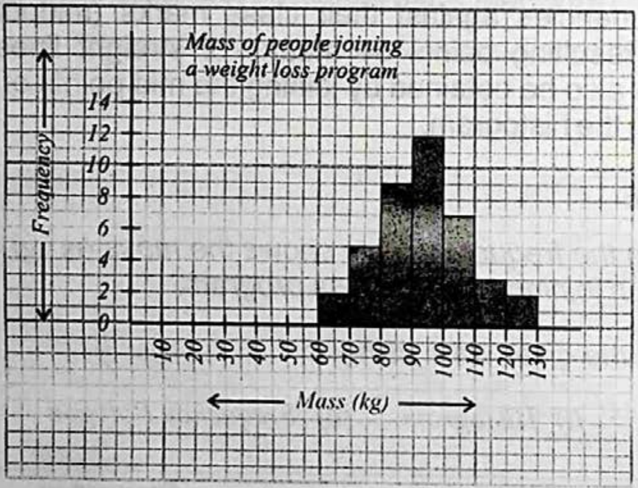
(iii) Label the horizontal axis mass (kg) and the vertical axis frequency.

(iv) Scale the horizontal and vertical axes.

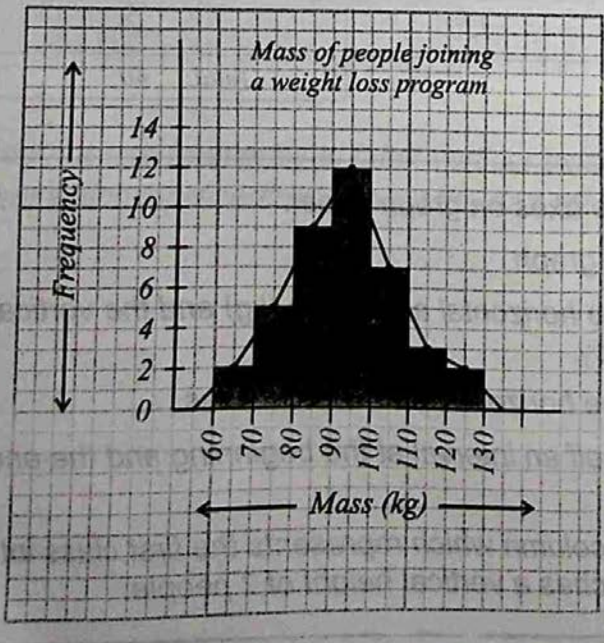
(v) Leave half an interval at the beginning and the end of the graph.

(vi) Draw a column which represents the first class interval and reaches a vertical height of 2 people.

Repeat step *3* for each of the other class intervals.



***(ii)* Mark the mid points of the tops of the rectangles obtained in the histogram from part *A*. Join the mid points by straight line intervals. Close the polygon by drawing lines which meet the horizontal axis a half column width before the first column and a half column width after the last column.**



EXERCISE – 10.1

1- Fifty Junior school children joined the school's computer club. Their ages were recorded.

10	8	9	10	7	8	8	11	10	9
7	8	9	9	10	11	11	10	9	8
8	7	9	7	10	7	10	8	9	11
10	11	8	10	9	8	9	7	11	10
9	10	10	11	10	11	7	11	10	9

Make a frequency table showing the number of each age and illustrate this information with a bar chart.

2- The local fish and chip shop had *56* customers on Saturday evening. They spent the following amount in rupees.

270	110	45	96	250	490	325	45
382	136	125	450	420	380	150	250
85	250	320	525	218	210	216	120
155	430	250	40	510	150	510	245
320	120	316	150	260	45	180	310
273	280	85	280	318	45	210	282
462	316	218	316	325	45	560	315

use groups *Rs.0-99, Rs.100-199, Rs.200-299, Rs.300-399, Rs.400-499, Rs.500-599* to make a frequency tables illustrate the data with a bar chart.

3- The weights to the nearest gram of *30* bags of popcorn sold at a festival are given as:

69	83	75	65	68	68	73	70	80	79
70	76	63	86	69	65	66	74	86	68
70	60	67	74	65	65	67	88	81	63

Make a frequency table, Illustrate the data with a bar chart.

10.2 CUMULATIVE FREQUENCY

The cumulative frequency shows the total number of observations (scores), which are below a certain value. We explain this with the help of following example:

EXAMPLE-1

All the students of class 9 took a maths test. Here are their marks as percentage. They have been grouped in 10. Find cumulative frequency.

Marks in %	Frequency
1 – 10	3
11 – 20	6
21 – 30	11
31 – 40	13
41 – 50	18
51 – 60	24
61 – 70	14
71 – 80	6
81 – 90	3
91 – 100	2
	Total Frequency: 100

(i) How many students scored *10 %* or less ?

(ii) How many students scored *20 %* or less ?

(iii) How many students scored *30 %* or less ?

(iv) How many students scored *40 %* or less ?

(v) How many students scored *50 %* or less?

(vi) How many students scored *60 %* or less?

(vii) How many students scored *70 %* or less?

(viii) How many students scored *80 %* or less?

(ix) How many students scored *90 %* or less?

(x) How many students scored *100 %* or less?

SOLUTION:

The following table shows cumulative frequencies.

Marks %	Cumulative Frequency	
10 % or less	3	3
20 % or less	9	$3+6=9$
30 % or less	20	$9+11=20$
40 % or less	33	$20+13=33$
50 % or less	51	$33+18=51$
60 % or less	75	$51+24=75$
70 % or less	89	$75+14=89$
80 % or less	95	$89+6=95$
90 % or less	98	$95+3=98$
100 % or less	100	$98+2=100$

EXAMPLE-2

The marks in a science test are given in the table. Complete the cumulative frequency.

Marks	Frequency	Marks	Cumulative frequency
1 – 10	5	10 or less	
11 – 20	6	20 or less	
21 – 30	8	30 or less	
31 – 40	16	40 or less	
41 – 50	23	50 or less	
51 – 60	18	60 or less	
61 – 70	12	70 or less	
71 – 80	10	80 or less	
81 – 90	2	90 or less	
91 – 100	0	100 or less	

SOLUTION:

Marks	Frequency	Marks	Cumulative Frequency
1 – 10	5	10 or less	5
11 – 20	6	20 or less	11
21 – 30	8	30 or less	19
31 – 40	16	40 or less	35
41 – 50	23	50 or less	58
51 – 60	18	60 or less	76
61 – 70	12	70 or less	88
71 – 80	10	80 or less	98
81 – 90	2	90 or less	100
91 – 100	0	100 or less	100

10.2.2 Cumulative Frequency Polygon

When the cumulative frequencies are plotted against the end points of their respective class intervals and joined together, the resultant graph is called a cumulative frequency polygon, or an **ogive**.

Therefore, an ogive can be considered as a line graph of the cumulative frequency results.

EXAMPLE

The data in the frequency table shows the number of fish caught by 28 competitors in fishing competition.

Construct a cumulative frequency polygon (that is, an ogive) for the given data.

Number of Fish Caught	Frequency
0 —— 5	3
5 —— 10	5
10 —— 15	6
15 —— 20	8
20 —— 25	4
25 —— 30	2
	Total: 28

SOLUTION:

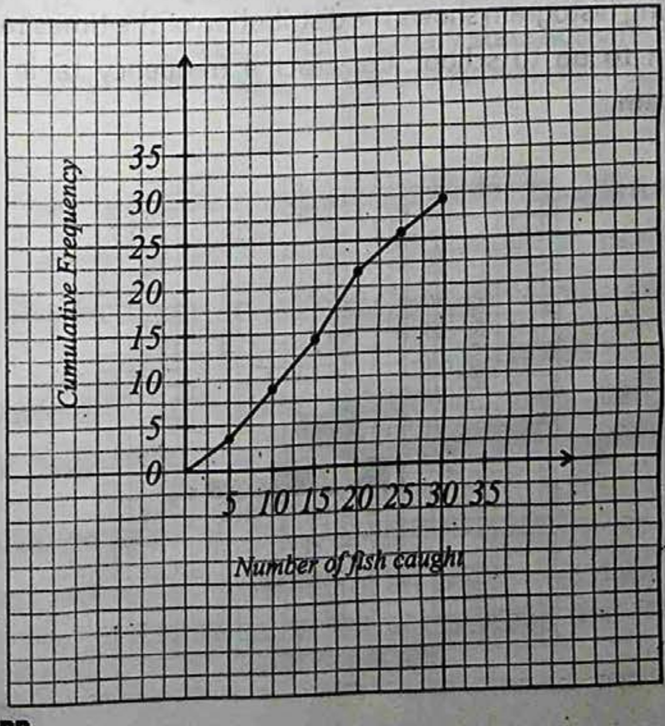
Find cumulative frequencies, draw the axes on graph paper, title the graph, label the horizontal axis "Number of fish caught" and the vertical axis "Cumulative frequency".

Number of fish caught	Frequency (f)	Cumulative Frequency (cf)
0 — 5	3	3
5 — 10	5	3 + 5 = 8
10 — 15	6	8 + 6 = 14
15 — 20	8	14 + 8 = 22
20 — 25	4	22 + 4 = 26
25 — 30	2	26 + 2 = 28
Total:	$\sum f = 28$	

Plot each of the cumulative frequencies against the end point of its respective class intervals.

The first point of an ogive curve has a cumulative frequency value of zero corresponding to the lowest possible value of the initial class interval. Therefore the first point of the ogive will be (0,0).

The next point will be (5,3) followed by (10,8) and so on. Connect each of the plotted points with straight line segments.



EXERCISE – 10.2

1- Draw a histogram to represent the frequency table in each of the following tables.

(i) The table shows the distribution of ages of *100* people attending a school function.

Age (years)	*0-19*	*20-39*	*40-59*	*60-79*	*80-89*
Frequency	*43*	*24*	*17*	*10*	*6*

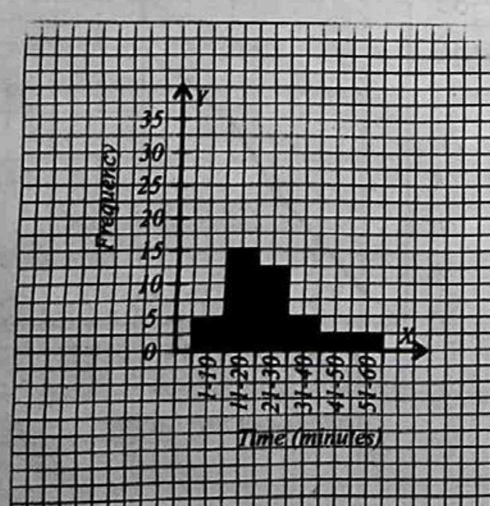
(ii) The table shows the results of a survey on the weekly earnings of *100* sixteen-year old boys.

Weekly earnings	*0-9*	*10-19*	*20-29*	*30-39*	*40-49*	*50-59*
Frequency	*45*	*10*	*11*	*21*	*10*	*3*

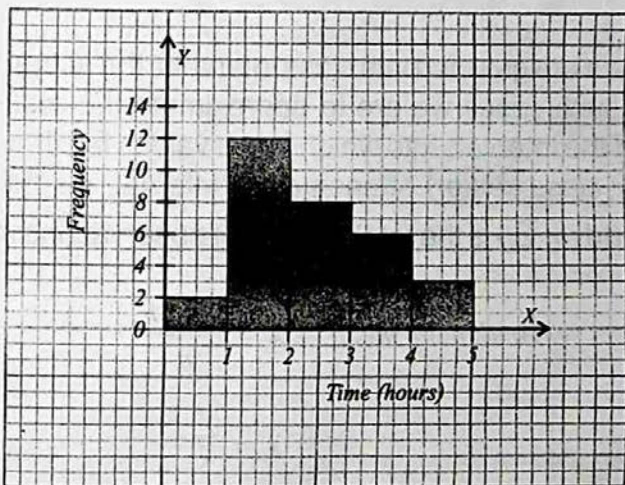
(iii) The table shows the distribution of the average marks of *40* children in the end-of-year examinations.

Average mark	*1-20*	*21-40*	*41-60*	*61-80*	*81-100*
Frequency	*2*	*4*	*19*	*12*	*3*

2- Following histogram shows the distribution of the times taken by *50* children to go to school. Construct a frequency table from the histogram.



3- Following histogram is based on the number of hours that *30* children spent watching television on a particular Saturday. Construct a frequency table from the histogram.



10.3 MEASURES OF CENTRAL TENDENCY

Measures of central tendency are summary statistics which measure the middle (or center) of the data. These are known as mean, median and mode.

(i) The mean is the average of all observations in a set of data.

(ii) The median is the middle observation in an ordered set of data.

(iii) The mode is the most frequent observation in a data set.

10.3.1 Calculate Arithmetic Mean or Mean, Median, Mode, Geometric Mean, Harmonic Mean from Ungrouped/Grouped Data

Arithmetic Mean by definition

By definition the Arithmetic mean of an ungrouped data is obtained by adding all numbers (scores) in the set together and then the total is divided by the number of scores in that set.

$$Arithmetic\ Mean = \frac{Sum\ of\ all\ scores}{Number\ of\ scores}$$

Symbolically this is written as: $\bar{x} = \frac{\sum x}{n}$, Where the symbol "$\Sigma$" indicates the "sum".

When the data are grouped into class intervals, the actual values *(or data)* are lost. In such cases we are to approximate the real values with the midpoints of the intervals into which these values fall. If 'x' represents the midpoint *(or class center)* of each class interval, f is the corresponding frequency and $(n=\Sigma f)$ is the total number of observations in a set. Then

$$Arithmetic\ mean = \bar{x} = \frac{\sum (fx)}{n}$$

$$\bar{x} = \frac{\sum fx}{\sum f}$$

EXAMPLE-1

The total monthly income of 8 persons are Rs.3000, Rs.4000, Rs.3500, Rs.4500, Rs.3800, Rs.4200, Rs.3600 and Rs.5400. Find their arithmetic mean.

SOLUTION: $x_1 = 3000,\quad x_2 = 4000,\quad x_3 = 3500,\quad x_4 = 4500,$

$x_5 = 3800,\quad x_6 = 4200,\quad x_7 = 3600,\quad x_8 = 5400,$

$$Arithmetic\ mean = \bar{x} = \frac{\sum x}{n}$$

$$= \frac{x_1 + x_2 + x_3 + x_4 + x_5 + x_6 + x_7 + x_8}{8}$$

$$= \frac{3000 + 4000 + 3500 + 4500 + 3800 + 4200 + 3600 + 5400}{8}$$

$$= \frac{32000}{8}$$

$$\bar{x} = Rs.4000$$

EXAMPLE-2

Find the arithmetic mean for the following distribution showing marks obtained by 50 students in mathematics in the annual exam.

Marks	20 – 24	25 – 29	30 – 34	35 – 39	40 – 44	45 – 49	50 – 54
Frequency	1	4	8	11	15	9	2

SOLUTION:

Marks	Frequency (f)	Class Mark (x)	fx
20 – 24	1	22	22
25 – 29	4	27	108
30 – 34	8	32	256
35 – 39	11	37	407
40 – 44	15	42	630
45 – 49	9	47	423
50 – 54	2	52	104

$$\bar{x} = \frac{\Sigma(fx)}{n} = \frac{22+108+256+407+630+423+104}{50}$$

$$= \frac{1950}{50}$$

$$= 39 \text{ marks}$$

Arithmetic Mean using Deviation from Assumed Means

The computation of mean using the formula $\bar{x} = \frac{\Sigma fx}{\Sigma f}$ is easy, provided the values of x_i and f_i are not large. If x_i and f_i are large we can save a considerable time taking deviations from assumed means. If 'A' is an assumed mean (which may be any number) and D_i denotes the deviations of X_i from 'A' i.e.

$$D_i = X_i - A, \text{ then } X_i = D_i + A, \text{ we have}$$

$$\overline{X} = A + \frac{\sum_{i=1}^{k} D_i}{n}$$

$$= A + \frac{\sum_{i=1}^{k} D}{n} \qquad \text{(ungrouped data)}$$

$$\text{and} \quad \overline{X} = A + \frac{\sum_{i=1}^{k} f_i D_i}{\Sigma f_i}$$

$$= A + \frac{\Sigma f D}{\Sigma f} \qquad \text{(grouped data)}$$

EXAMPLE-1

Find the arithmetic mean from the following values using formula:

$$\overline{X} = A + \frac{\Sigma D}{n}$$

184, 191, 172, 168, 187,189, 196, 186, 193, 195.

SOLUTION:

Taking deviation from the assumed mean $A=180, D=X-A$, we get

$$D = 184-180,\ 191-180,\ 172-180,\ 168-180,\ 187-180,\ 189-180,$$
$$196-180,\ 189-180,\ 193-180,\ 195-180$$

$$= 4,\ 11,\ -8,\ -12,\ 7,\ 9,\ 16,\ 9,\ 13,\ 15$$

$$\overline{X} = A + \frac{\Sigma D}{n} = 180 + \frac{4+11-8-12+7+9+16+9+13+15}{10}$$

$$= 180 + \frac{64}{10}$$

$$= 180 + 6.4$$

$$= 186.4$$

EXAMPLE-2

Find the mean weight of 120 students for the distribution of weights in the following table. Using formula:

$$\overline{X} = A + \frac{\Sigma f D}{\Sigma f}$$

Weight (Pounds)	Class Mark (X)	Frequency (f)	$D = X - 144.5$	fD
110 – 119	114.5	1	– 30	– 30
120 – 129	124.5	4	– 20	– 80
130 – 139	134.5	17	– 10	– 170
140 – 149	144.5 ← A	28	0	0
150 – 159	154.5	25	10	250
160 – 169	164.5	18	20	360
170 – 179	174.5	13	30	390
180 – 189	184.5	6	40	240
190 – 199	194.5	5	50	250
200 – 209	204.5	2	60	120
210 – 219	214.5	1	70	70
		$n = \Sigma f = 120$	$\Sigma f D = 1680 - 280 = 1400$	

SOLUTION: $\overline{X} = A + \frac{\Sigma f D}{\Sigma f}$

$$= 144.5 + \frac{1400}{120}$$

$$= 144.5 + 1167$$

$$= 156.17 \text{ pounds}$$

Median (ungrouped data)

The median is middle value of any set of data arranged in numerical order. In the set of n numbers, the median is located at the $\frac{n+1}{2}$ *th* score.

The median is:

- the middle score for an odd number of scores arranged in numerical order.
- the average of the two middle scores for an even number of scores arranged in numerical order.

EXAMPLE

For the data set 6, 2, 4, 3, 4, 5, 4, 5. Find the median.

SOLUTION:

$2\ 3\ 4\ 4\ 4\ 5\ 5\ 6$ *(Check the scores are arranged in numerical order)*

$2\ 3\ 4\ 4\ 4\ 5\ 5\ 6$ *{Locate the position of the median using the rule $\frac{n+1}{2}$ where $n=8$. This places the median as the $4.5th$ score $= 4th$ score $- 0.5$ ($5th$ score $- 4th$ score)}*

$$Median = \frac{4+4}{2}$$

$$= \frac{8}{2}$$

$$= 4$$ *(obtain the average of two middle scores)*

The median is 4.

Median (grouped data)

The median is obtained by drawing an ogive of the data and estimating the median from the $50th$ percentage.

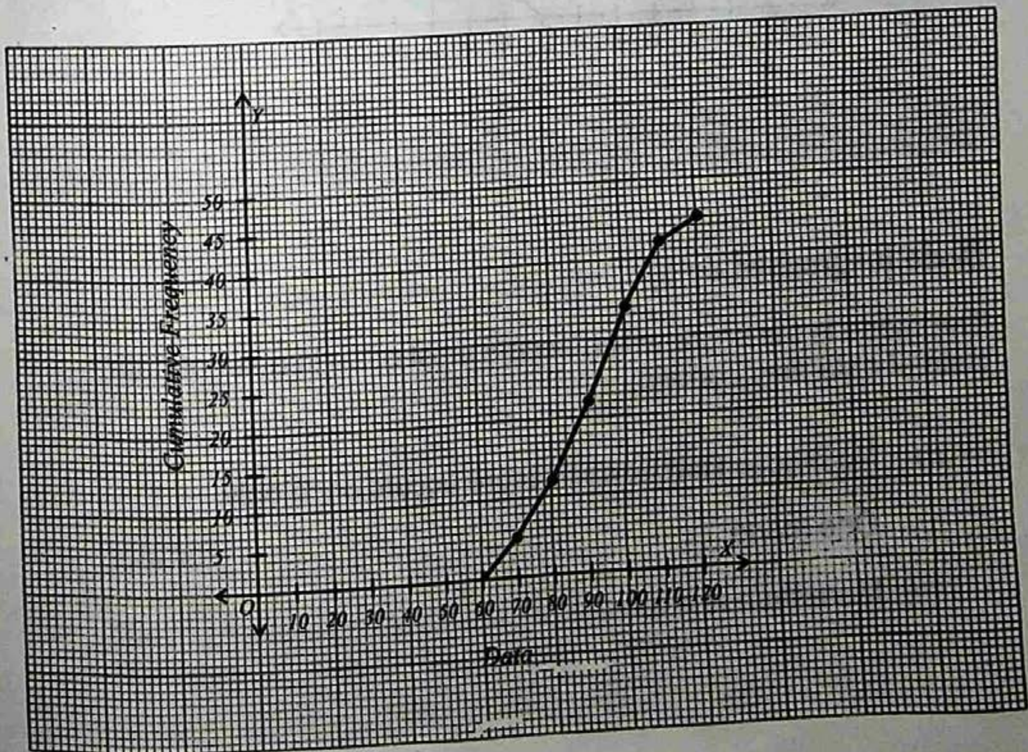
EXAMPLE

For the given data estimate the median.

Class interval	Frequency (*f*)	Cumulative Frequency (*Cf*)
60 – 70	5	5
70 – 80	7	12
80 – 90	10	22
90 – 100	12	34
100 – 110	8	42
110 – 120	3	45
Total	45	

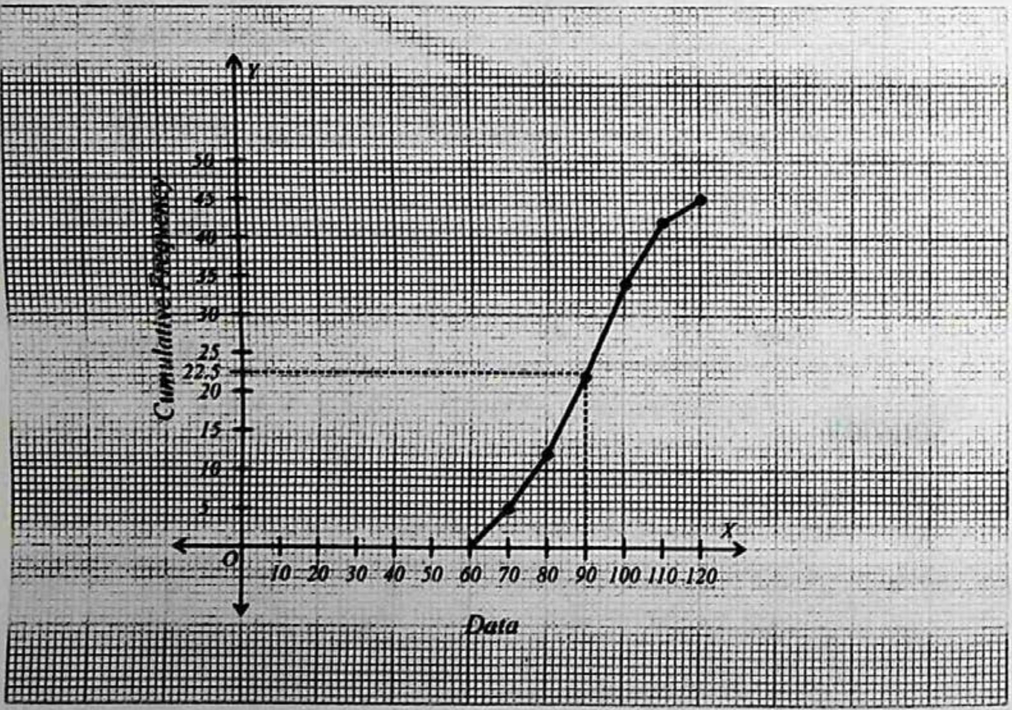
SOLUTION:

Draw the ogive.



Locate the middle of the cumulative frequency axis, that is 22.5 and label it.

Draw a horizontal line from the point to the ogive and then vertically to the horizontal axis. Read the value of the median from the x-axis. The median for the given data is approximately 90.



Mode (ungrouped data)

The mode is the score which occurs most often in a set of data. Sets of data may contain:

(i) No mode: That is, each score occurs once only.

(ii) One mode

(iii) More than one mode

EXAMPLE

For the data set 6, 2, 4, 3, 4, 5, 4, 5, find the mode.

SOLUTION:

We work through the set and see that value in the data which occurs the greatest number of times. The mod is 4.

$$\begin{array}{cccccccc} & & & & & \downarrow & \downarrow & \\ 2 & 3 & 4 & 4 & 4 & 5 & 5 & 6 \\ & & \uparrow & \uparrow & \uparrow & & & \end{array}$$

Mode (grouped data)

We do not find a mode, because exact scores are lost. We can however, find a model class. This is the class interval that has the highest frequency.

EXAMPLE

Find the model class from the following table.

Class interval	Frequency
60 – 70	*5*
70 – 80	*7*
80 – 90	*10*
90 – 100	*12*
100 – 110	*8*
110 – 120	*3*
Total	*45*

SOLUTION:

The model class is the class interval with the highest frequency.
The model class is the 90 – 100 class interval.

Geometric Mean (ungrouped data)

The geometric mean "G" of a set of n positive values $x_1, x_2, ..., x_n$ is the nth root of the product of the values. Thus

$$G = \sqrt[n]{x_1.x_2.x_3 \cdots, x_n}$$

$$= (x_1.x_2.x_3 \cdots, x_n)^{\frac{1}{n}}$$

EXAMPLE

Geometric mean of 2,4 and 8 is:

SOLUTION: $G = \sqrt[3]{2 \times 4 \times 8} \quad = \sqrt[3]{64}$

$$= [(4)^3]^{\frac{1}{3}}$$

$$= 4$$

Geometric Mean (grouped data)

Let $x_1, x_2, ..., x_k$ represent the class marks in a frequency distribution with $f_1, f_2, ..., f_k$ as the corresponding class frequencies (where $f_1 + f_2 + ... + f_k = \Sigma f = n$). Since x_1 occurs f_1 times and so on. x_k occurs f_k times, then the product of original values may be written as :

$$\underbrace{x_1.x_1....x_1}_{f_1\ times} \quad \underbrace{x_2.x_2....x_2}_{f_2\ times} \quad ... \quad \underbrace{x_k.x_k....x_k}_{f_k\ times}$$

or $x_1^{f_1}.\ x_2^{f_2} ...\ x_k^{f_k}$ and the geometric mean is:

$$G = \sqrt[n]{x_1^{f_1}.\ x_2^{f_2} ...\ x_k^{f_k}}$$

$$= (x_1^{f_1}.\ x_2^{f_2} ...\ x_k^{f_k})^{1/n}$$

This is sometimes called the weighted geometric mean with weights $f_1, f_2, \ldots, f_k$.

EXAMPLE

Find the geometric mean for the following frequency distribution.

x	1	2	3	4
f	2	3	4	1

SOLUTION:

Here $\Sigma f = 2+3+4+1=10$

$$G = (x_1^{f_1} . x_2^{f_2} \ldots x_k^{f_k})^{1/n}$$

$$= ((1)^2 . (2)^3 . (3)^4 . (4)^1)^{1/10}$$

$$= (1\times 8\times 81\times 4)^{1/10}$$

$$= (2592)^{1/10} = 2.1946$$

Harmonic Mean (ungrouped data)

The harmonic mean H of a set of n values $x_1, x_2, \cdots, x_n$ is the reciprocal of the arithmetic mean of the reciprocal of the values.

$$H = \frac{n}{\frac{1}{x_1}+\frac{1}{x_2}+\cdots+\frac{1}{x_n}} = \frac{n}{\sum_{i=1}^{n}\left(\frac{1}{x_i}\right)} = \frac{n}{\Sigma\left(\frac{1}{x}\right)}$$

EXAMPLE

Find the harmonic mean of the values 2,4 and 8.

SOLUTION: $H = \dfrac{3}{\frac{1}{2}+\frac{1}{4}+\frac{1}{8}} = \dfrac{3}{\frac{4+2+1}{8}} = \dfrac{3}{\frac{7}{8}} = \dfrac{3\times 8}{7} = \dfrac{24}{7} = 3.43$

The Harmonic Mean (grouped data)

Let $x_1, x_2, ...x_k$ represent the class marks and $f_1, f_2, \cdots f_k$ as the corresponding class frequencies (where $f_1 + f_2 + \cdots + f_k = \Sigma f_i = n$).

Then the reciprocal of the class marks will be $\frac{1}{x_1}, \frac{1}{x_2}, \cdots, \frac{1}{x_k}$. Since the reciprocals occur with frequencies $f_1, f_2, \cdots f_k$. The total value of the reciprocals in the first class is $\frac{f_1}{x_1}$ in the second class is $\frac{f_2}{x_2}, \cdots$ in the k^{th} class is $\frac{f_k}{x_k}$. Then $\frac{f_1}{x_1} + \frac{f_2}{x_2} + \cdots + \frac{f_k}{x_k} = \sum_{i=1}^{k} \frac{f_i}{x_i} = \Sigma f(\frac{1}{x})$.

The harmonic mean is defined as:

$$H = \frac{\sum_{i=1}^{k} f_i}{\sum_{i=1}^{k} f_i(\frac{1}{x_i})} = \frac{\Sigma f}{\Sigma f(\frac{1}{x})} = \frac{n}{\Sigma f(\frac{1}{x})}$$

Sometimes this is also called weighted harmonic mean with weights $f_1, f_2, \cdots f_k$.

10.3.2 Properties of Arithmetic Mean

Following are the properties of Arithmetic mean.

(i) The sum of deviations of values from their mean is zero, symbolically:

$$\Sigma (x_i - \overline{X}) = 0 \text{ or } \Sigma f_i (x_i - \overline{X}) = 0$$

(ii) If n_1 values have mean $\overline{x_1}$, n_2 values have mean $\overline{x_2}, \cdots, n_k$ values have mean $\overline{x_k}$, then mean of all the values is:

$$\overline{X} = \frac{n_1\overline{X_1} + n_2\overline{X_2} + \cdots + n_k\overline{X_k}}{n_1 + n_2 + \cdots + n_k}$$

$$= \frac{\sum_{i=1}^{k} n_i\overline{X_i}}{\sum_{i=1}^{k} n_i}$$

(iii) The sum of squares of the deviations of the values x_i from any value "a" is minimum if and only if $a = \overline{X}$, symbolically $\Sigma(x_i - a)^2$ is a minimum if and only if $a = \overline{X}$.

10.3.3 Weighted Mean and Moving Average

If $x_1, x_2, \cdots, x_k$ have weights $w_1, w_2, \cdots, w_k$, then the weighted arithmetic mean or the weighted mean (denoted by $\overline{x_w}$) is defined as:

$$\overline{X_w} = \frac{w_1x_1 + w_2x_2 + \cdots + w_kx_k}{w_1 + w_2 + \cdots + w_k}$$

$$= \frac{\sum_{i=1}^{k} w_ix_i}{\sum_{i=1}^{k} w_i}$$

Moving Average

The average calculated by using *"n"* consecutive values of the observed series, for example we have to calculate *3* year moving average, then we take first three values from the series, add them and place against the middle of its time period. Then repeat the operation by dropping 1st value from the beginning and including first value after the preceding total. Mathematical form:

$$a_1 = \frac{1}{3}(y_1 + y_2 + y_3)$$

$$a_2 = \frac{1}{3}(y_2 + y_3 + y_4) \text{ and soon.}$$

Quartiles

We know that the median of an array is the middle value (or the mean of the two middle values). It divides a set of data into two equal parts. There are certain other values which divided a set of data into four equal parts called as first, second and third quartiles. These are denoted by Q_1, Q_2 and Q_3 respectively.

The first and third quartiles are also called as lower and upper quartiles respectively. The second quartile is the median.

$$Q_1 = \text{value of} \left(\frac{n+1}{4}\right) \text{th item.}$$

$$Q_2 = \text{value of } 2\left(\frac{n+1}{4}\right) \text{th item or} \left(\frac{n+1}{2}\right) \text{th item.}$$

$$Q_3 = \text{value of } \frac{3(n+1)}{4} \text{th item.}$$

EXAMPLE

Find all the quartiles from the following marks obtained by 20 students in statistic test.

53	74	82	42	39	20	81	68	58	28
67	54	93	70	30	55	36	37	29	61

SOLUTION:

The marks of $n=20$ students are arranged in ascending order as follows:

20	28	29	30	36	37	39	42	53	54
55	58	61	67	68	70	74	81	82	93

$Q_1 =$ value of $\left(\frac{n+1}{4}\right)th$ or $\left(\frac{20+1}{4}\right)th$ or $5.25th$ item from below.

The value of the $5th$ item is 36 and that of the $6th$ item is 37. Thus the first quartile is a value of $0.25th$ of the way between 36 and 37, which is 36.25.

$$\text{Thus } Q_1 = 36.25$$

$\underset{(median)}{Q_2} =$ value of $2\left(\frac{n+1}{4}\right)th$ or $\frac{2(20+1)}{4}th$ or $\frac{21}{2}th$ or $10.5th$ item from below.

The value of the $10th$ item is 54 and that of the $11th$ item is 55. Thus the second quartile is a value of $0.5th$ of the way between 54 and 55, which is 54.5.

$$\text{Thus } Q_2 = 54.5$$

$Q_3 =$ value of $\frac{3(n+1)}{4}th$ or $\frac{3(20+1)}{4}th$ or $\frac{3\times 21}{4}th$ or $\frac{63}{4}th$ or $15.75th$ item from below.

The value of the $15th$ item is 68 and that of the $16th$ item is 70. Thus the first quartile is a value of $0.75th$ of the way between 68 and 70, which is $68+2(0.75)=69.5$.

$$\text{Thus } Q_3 = 69.5$$

EXAMPLE

The marks of 100 students in an examination are given in the following table.

Marks	No.of Students	Cumulative Frequency
1 —— 10	2	2
11 —— 20	12	14
21 —— 30	25	39
31 —— 40	29	68
41 —— 50	15	83
51 —— 60	10	93
61 —— 70	4	97
71 —— 80	3	100

Use ogive to estimate the quartile, that is

(i) The upper quartile.

(ii) The lower quartile.

$\frac{3}{4}$ of the total frequency

$$=\frac{3}{4}\times 100 = 3\times 25 = 75$$

From the curve the upper quartile = 44

$\frac{1}{4}$ of the total frequency

$$=\frac{1}{4}\times 100 = 25$$

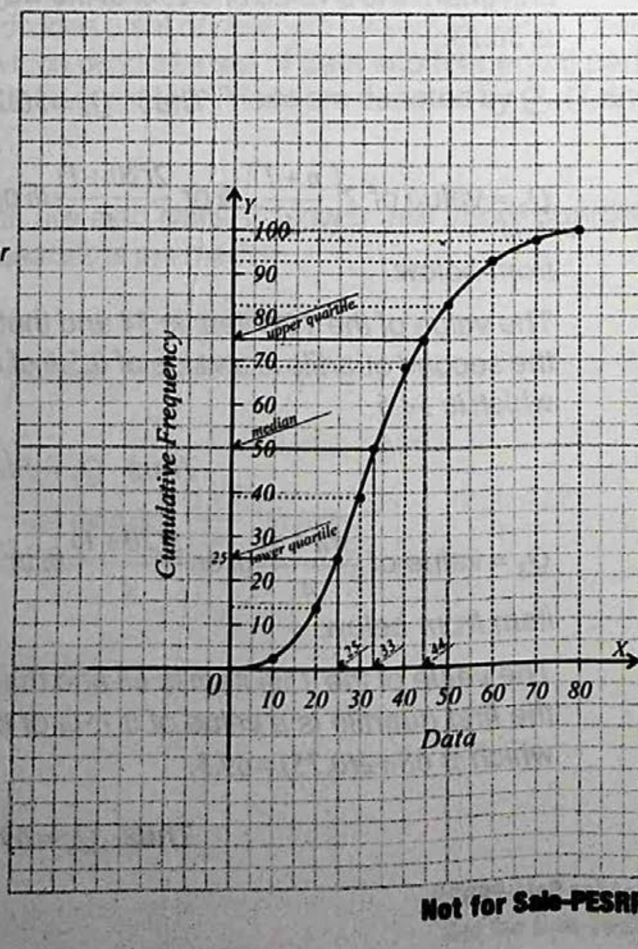
From the curve the lower quartile = 25

$\frac{1}{2}$ of the total frequency

$$=\frac{1}{2}\times 100 = 50$$

From the curve the median mark = 33

Two small squares along x-axis represents 10 units.
Two small squares along y-axis represents 10 units.



10.3.4 Estimate Median, Quartile and Mode Graphically, Graphic Location of Median

The approximate value of the median can be located from an ogive (a cumulative frequency polygon). In an ogive, the median is the value of x corresponding to $\frac{n}{2}$. Thus to locate median, we mark $\frac{n}{2}$ along the y-axis and draw a perpendicular from this point of y-axis and extend it so as to intersect the ogive. Then we drop a perpendicular on th x-axis from the point of intersection. The point at which the perpendicular intersects the x-axis is the value of the median.

For the given data estimate the median, Mode

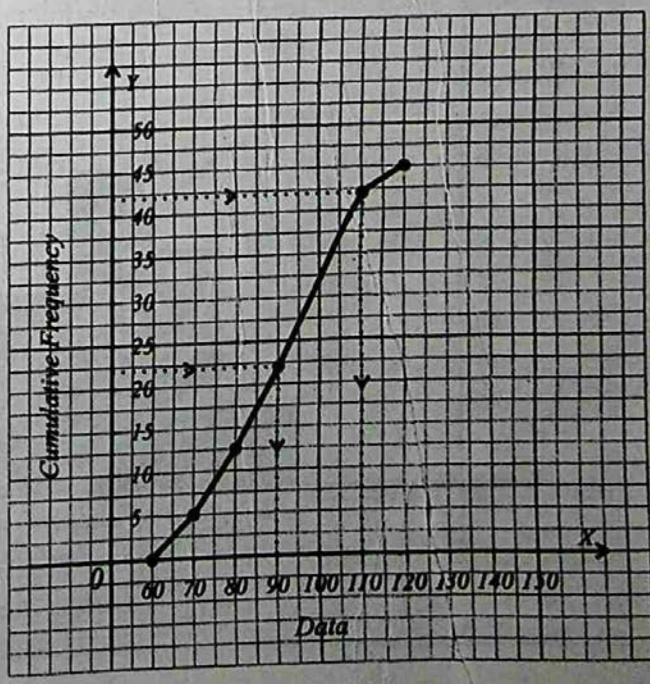
Class-intervals	Frequency	Cumulative Frequency
60 — 70	5	5
70 — 80	7	12
80 — 90	10	22
90 — 100	12	34
100 — 110	8	42
110 — 120	3	45
	45	—

(i) Draw the ogive

(ii) Locate the middle of the cumulative frequency axis, that is 22.5 and label it.

(iii) Draw a horizontal line from this point to the ogive and then vertically to the horizontal axis.

(iv) Read the value of the median from the x-axis. The median for the given data is approximately *90*.



EXAMPLE

Calculate mode by graphic method from the given data.

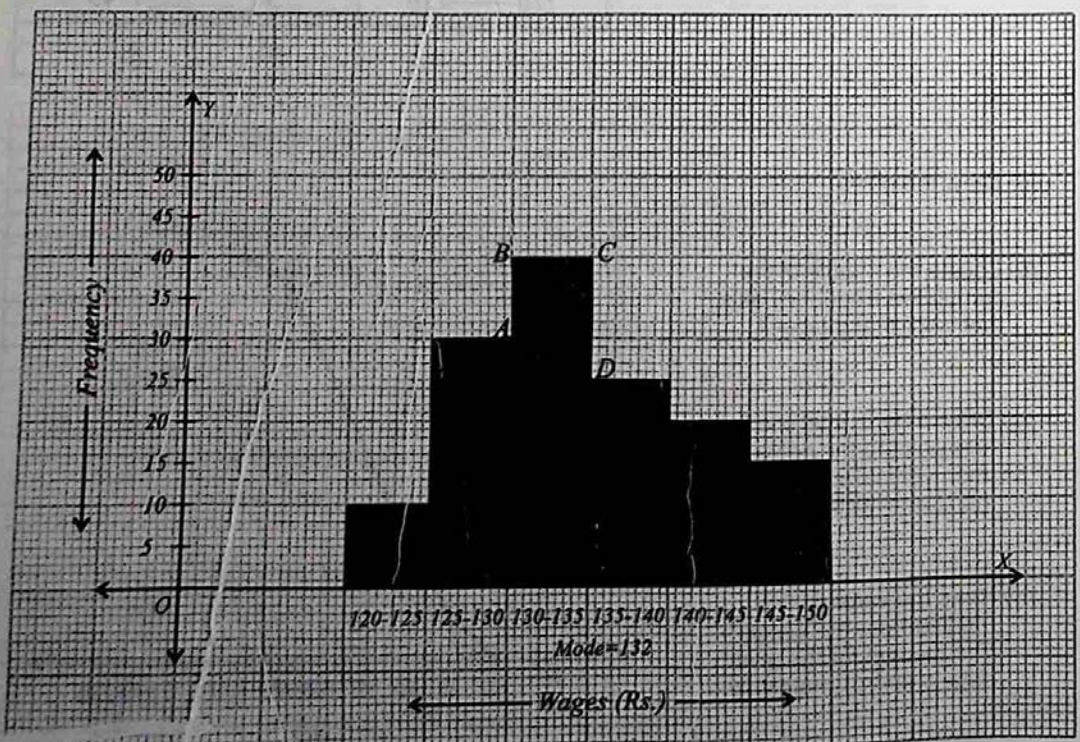
Wages Rs.	120 – 125	125 – 130	130 – 135	135 – 140	140 – 145	145 – 150
f	10	30	40	25	20	15

SOLUTION:

Wages in rupees are taken along x-axis while the frequency along the y-asix. One big square along x-axis represents 5units while one big square along y-axis represents 10units. We join the extreme ends of the bar with maximum frequency 30 and 25 as shown in the figure.

The two line-segments AC and BD cut each other at point P. We drop a perpendicular from P which meets x-axis at M. The value of M is 132.

Thus Mode = 132.



Graphic Location of Quartiles

EXAMPLE

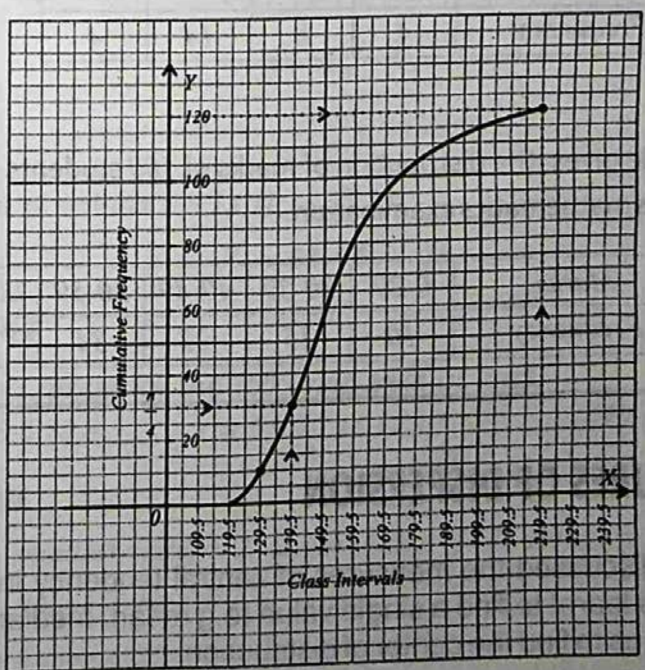
The following table shows a frequency distribution of grades on a final examination in mathematics. Locate quartiles graphically.

Grade	90-99	80-89	70-79	60-69	50-59	40-49	30-39
No of Students	9	32	43	21	11	3	1

SOLUTION:

For the graphic location of quartile, students are shown in the following table. We write the given data as follow:

Grade	Number of Students (f)	Class Intervals	Cumulative Frequency
30-39	1	29.5 – 39.5	1
40-49	3	39.5 – 49.5	4
50-59	11	49.5 – 59.5	15
60-69	21	59.5 – 69.5	36
70-79	43	69.5 – 79.5	79
80-89	32	79.5 – 89.5	111
90-99	9	89.5 – 99.5	120



EXERCISE – 10.3

1- Represent the given data using Frequency polygon.

(i) The table shows the distribution of marks of *30* children in a test.

Marks	*0-39*	*40-49*	*50-79*	*80-99*
Frequency	*8*	*8*	*10*	*4*

(ii) The table shows the distribution of time (in seconds) taken for *40* children to complete the obstacle race.

Time (second)	*1-40*	*41-50*	*51-60*	*61-70*
Frequency	*8*	*15*	*7*	*10*

(iii) The table shows the distribution of weights of *30* bags of chips from a fish and chip shop.

Weights (grams)	*1-50*	*51-60*	*61-70*	*71-80*
Frequency	*4*	*8*	*14*	*4*

(iv) The table gives the distribution of marks of *100* students in an end-of-terms mathematics examination.

Marks	*0-29*	*30-39*	*40-49*	*50-59*	*60-99*
Frequency	*10*	*15*	*25*	*34*	*16*

10.4 MEASURES OF DISPERSION

We have discussed the measures of central tendency in the preceding section, we learnt to find a single value (e.g mean, median, mode) which would help us to find the center of distribution.

For this let us consider the following distributions.

1.	63, 63, 63, 63, 63, 63, 63, 63	$\overline{X} = 63$
2.	62, 62, 62, 63, 63, 64, 64, 64	$\overline{X} = 63$
3.	48, 49, 57, 63, 69, 68, 74, 78	$\overline{X} = 63$
4.	40, 41, 47, 52, 62, 87, 88, 94	$\overline{X} = 63$

The above mentioned distributions have the same mean i:e *63*, but these distributions differ greatly in their dispersion, i.e. the extent to which the values are spread out from the average.

Though there is a great difference in the dispersion of the values of the distribution, yet each of these distributions is described by the same mean, i.e. *63*. We therefore need a measure to see how dispersed the data is.
The measures used for this purpose are called "Measures of Dispersion".

Range

The simplest measure of dispersion is range. Range is the difference between the largest value and the smallest value in the data. If the smallest value is denoted by X_o and the largest value is denoted by X_m then the range denoted by R is given by:

$$R = X_m - X_o$$

For Example

For the set of values *6,8,13,11,18,27,23*

$$R = 27 - 6 = 21$$

EXAMPLE

Find the range for the following sets of data.

(i) 2,9,3,6,12,8 *(ii) 6,8,10,14,13,12* *(iii) 10,18,9,6,22*

SOLUTION:

(i) smallest value = 2, largest value = 12, Range = 12 − 2 = 10

(ii) Range = 14 − 6 = 8

(iii) Range = 22 − 6 = 16

Variance

Variance is defined as the square of the standard deviation, i.e. the mean of the squared deviation from the mean. It is given by:

$$Var = \frac{\sum(x-\overline{X})^2}{n} \text{(for ungrouped data)}$$

$$Var = \frac{\sum f(x-\overline{X})^2}{\sum f} \text{(for grouped data)}$$

Standard Deviation

It is defined as the positive square root of the mean of the squared deviations of the values from their mean, The standard deviation of a set of n values, $x_1, x_2, \ldots\ldots. x_n$ is denoted by S, where:

$$S = \sqrt{\frac{\sum(x-\overline{X})^2}{n}} \text{(for ungrouped data)}$$

$$S = \sqrt{\frac{\sum f(x-\overline{X})^2}{\sum f}} \text{(for grouped data)}$$

$$\text{where } n = f_1 + f_2 + \cdots + f_k = \sum f$$

EXAMPLE-1

Find the standard deviation for the values, 1,2,3,4,6,8,11. Also find the variance in this case.

SOLUTION: Here $\overline{X} = \frac{1+2+3+4+6+8+11}{7} = \frac{35}{7} = 5$

$$S = \sqrt{\frac{\sum(x-\overline{X})^2}{n}}$$

$$= \sqrt{\frac{(1-5)^2+(2-5)^2+(3-5)^2+(4-5)^2+(6-5)^2+(8-5)^2+(11-5)^2}{7}}$$

$$= \sqrt{\frac{16+9+4+1+1+9+36}{7}} = \sqrt{\frac{76}{7}}$$

$$= \sqrt{10.86} = 3.295$$

Thus standard deviation = $S = 3.63318$

$$\text{Variance} = \frac{\Sigma(x-\overline{X})^2}{n} = \frac{76}{7}$$

$$= 10.86$$

EXAMPLE-2

Find the standard deviation for the frequency distribution of marks obtained by 50 students in English at a certain examination. Also find the variance in this case.

Marks	20–24	25–29	30–34	35–39	40–44	45–49	50–54
Frequency	1	4	8	11	15	9	2

SOLUTION: Here

Marks	Frequency (f)	Class Mark (X)	fx	$x-\overline{X}$	$(x-\overline{X})^2$	$f(x-\overline{X})^2$
20–24	1	22	22	− 17	289	289
25–29	4	27	108	− 12	144	576
30–34	8	32	256	− 7	49	392
35–39	11	37	407	− 2	4	44
40–44	15	42	630	3	9	135
45–49	9	47	423	8	64	576
50–54	2	52	104	13	169	338
	$n=\Sigma f=50$	$\Sigma f=259$	$\Sigma fx=1950$			2350

$$\overline{X} = \frac{\Sigma f x}{\Sigma f} = \frac{1950}{50} = 39 \text{ marks}$$

$$S = \sqrt{\frac{\Sigma f(x-\overline{X})^2}{n}} = \sqrt{\frac{2350}{50}} = \sqrt{47}$$

Thus standard deviation = S.D = 6.85

$$\text{Variance} = \frac{\Sigma f(x-\overline{X})^2}{n} = \frac{2350}{50}$$

$$= 47$$

EXERCISE – 10.4

1- Construct a cumulative frequency polygon (that is, an ogive) for the given data.

(i) The table shows the distribution of weights (in kilograms) of 60 boys of ten years of age.

Weight (kg)	31-36	37-39	40-42	43-45	46-54
Frequency	8	10	18	12	12

(ii) The table shows the distribution of times taken (in minutes) for 50 children of five years age to eat their school dinners.

Time (minutes)	4-5	6-7	8-9	10-11	12-15	16-19	20-29
Frequency	5	4	10	9	6	6	10

(iii) The table shows the distribution of the ages of people boarding buses at the bus station between 08.30 to 09.00 in the morning.

Age (years)	0-9	10-19	20-29	30-39	40-69
Frequency	10	20	30	20	15

(iv)

Classes	5-10	10-15	15-20	20-25	25-30
Frequency	10	15	20	30	15

(v) The table gives the distribution of weights (kilograms) of 100 people.

Weight (kilograms)	50-59	60-69	70-79	80-89	90-99	100-109
Frequency	15	30	35	15	3	2

Review Exercise – 10

1- Encircle the correct answer.

(i) When a bar graph is constructed, so that the area of each bar is proportional to the number of items in each group is called.

(a) *curve* (b) *ogive* (c) *histogram* (d) *bar diagram*

(ii) The summary statistics which measure the middle (or center) of the data is called:

(a) *mean* (b) *mode*
(c) *median* (d) *all of these*

(iii) If all numbers in a set are added together and then the total is divided by the number of scores in the set is called

(a) *mean* (b) *mode* (c) *median* (d) *weighted mean*

(iv) The middle values of data arranged in numerical order is called

(a) *mode* (b) *median* (c) *mean* (d) *geometric mean*

(v) The score which occurs most often in a set of data is called

(a) *mode* (b) *mean* (c) *median* (d) *geometric mean*

(vi) $\overline{X} = \dfrac{x_1 + x_2 + x_3 + \cdots + x_n}{n}$ is called

(a) *mean value of* $x_1, x_2, \cdots, x_n$ (b) *arithmetic mean*
(c) *geometric mean* (d) *weighted mean*

(vii) $H = \dfrac{n}{\sum\left(\dfrac{1}{x}\right)}$ is called

(a) *harmonic mean* (b) *mode* (c) *mean* (d) *arithmetic mean*

(viii) $\overline{X}_w = \dfrac{\sum wx}{\sum w}$ is called

(a) *arithmetic mean* (b) *weighted mean*
(c) *geometric mean* (d) *mean*

(ix) $\sum(x_i - \overline{X}) = 0$ is one of the properties of

(a) *arithmetic mean* (b) *geometric mean*
(c) *harmonic mean* (d) *mode*

2- Fill in the blanks.

(i) When a bar graph is constructed, so that the area of each bar is proportion to the number of items in each group is called a ______________.

(ii) The summary statistic which measure the middle (or center) for the data is called ______________.

(iii) If all numbers in a set are added together and then the total is divided by the number of scores in the set is called ______________.

(iv) The middle value of data arranged in numerical order is called ______________.

(v) The score which occurs most often in a set of data is called ______________.

(vi) $\overline{X} = \dfrac{x_1 + x_2 + x_3 + \cdots + x_n}{n}$ is called the ______________.

(vii) The *nth* root of the product of the values of a set of *n* positive values is called ______________.

(viii) $H = \dfrac{n}{\sum\left(\dfrac{1}{x}\right)}$ is called the ______________.

(ix) $\overline{X}_w = \dfrac{\sum wx}{\sum w}$ is called the ______________.

(x) $\sum(x_i - \overline{X}) = 0$ is one of the properties of ______________.

3- Find the standard deviation of the values *2, 3, 6, 8, 11.*

4- Find the standard deviation and variance for a set of ungrouped values, when $n = 15$, $\sum(x - \overline{x})^2 = 1444$

5- For the data *3, 5, 6, 8, 8, 9, 10* , find
(i) Mean (ii) Median (iii) Mode

6- Find the mean, median and mode for the set of the value
4, 6, 7, 4, 8, 9,, 7, 10.

SUMMARY

- When a bar chart is construed, so that the area of each bar is proportional to the number of items in each group is called a histogram.

- Cumulative frequency is a running total of class frequencies.

- When the cumulative frequencies are plotted against the end point of their respective class intervals and joined together, the resultant graph is called a cumulative frequency polygon or ogive.

- These are summary statistics which measure the middle (or center) of the data.

- To obtain the mean of a data, all numbers in the set are added together and then the total is divided by the number of scores in that set.

- The middle value of data arranged in numerical order is called median.

- The mode is the score which occurs most often in a set of data.

- Arithmetic mean of an ungrouped data is obtained by adding all numbers (scores) in the set together and then the sum is divided by the number of scores in that set.

- The geometric mean *"G"* of a set of n positive values $x_1, x_2, \ldots, x_n$ is the nth root of the product of the values.

- The harmonic mean H of a set of n values $x_1, x_2, \cdots, x_n$ is the reciprocal of the arithmetic mean of the reciprocal of the values.

SUMMARY

- Variance is defined as the square of the standard deviation, i.e. the mean of the squared deviation from the mean.

- The simplest measure of dispersion is range. Range is the difference between the largest value and the smallest value in the data.

- The average calculated by using "n" consecutive values of the observed series, for example we have to calculate 3 years moving average, then we take first three values from the series, add them and place against the middle of its time period. Then repeat the operation by dropping 1st value from the beginning and including first value after the preceding total.

- Standard deviation is defined as the positive square root of the mean of the squared deviations of the values from their mean, The standard deviation of a set of n values, $x_1, x_2, \ldots\ldots x_n$ is denoted by S.

- If $x_1, x_2, \cdots, x_k$ have weights $w_1, w_2, \cdots, w_k$, then the weighted arithmetic mean or the weighted mean (denoted by $\overline{x_w}$) is defined as:

$$\overline{x_w} = \frac{\sum_{i=1}^{k} w_i x_i}{\sum_{i=1}^{k} w_i}$$

Answers

Exercise 1.1

1- (i) $\frac{19}{20}$ (ii) $\frac{13}{20}$ (iii) $\frac{3}{4}$ (iv) $\frac{1}{4}$ (v) $\frac{14}{25}$ (vi) $\frac{12}{25}$ (vii) $\frac{2}{25}$ (viii) $\frac{67}{200}$
(ix) $\frac{3}{8}$ (x) $\frac{7}{8}$ (xi) $\frac{21}{400}$ (xii) $\frac{17}{40}$

2- (i) 75% (ii) 60% (iii) 16% (iv) 65% (v) 124% (vi) 52.5% (vii) 38.3%
(viii) 266.66% (ix) 160% (x) 87.5% (xi) 62.5% (xii) 37.5%

3- (i) 0.47 (ii) 0.58 (iii) 0.92 (iv) 0.08 (v) 0.12 (vi) 1.20
(vii) 1.80 (viii) 1.45 (ix) 0.055 (x) 0.0533 (xi) 0.486 (xii) 0.583

4- (i) 50% (ii) 90% (iii) 125% (iv) 139% (v) 172% (vi) 22%
(vii) 264% (viii) 341% (ix) 84.5% (x) 178% (xi) 158% (xii) 6.5%

5- (ii) 80% (iii) $\frac{2}{5}$, 0.4 (iv) $\frac{31}{50}$, 62% (v) $\frac{11}{25}$, 0.44

Exercise 1.2

1- 55% 2- 18% 3- 12% 4- (i) $P = 208, C = 68, I = 48, B = 76$ (ii) 19%

5- Math 6- 27.78% 7- 500 pages 8- Rs.7500 9- 20%

Exercise 1.3

1- (i) 4 : 1 (ii) 4 : 1 (iii) 1 : 4 (iv) 1 : 1 (v) 5 : 4 (vi) 6 : 11

2- (i) 10 : 9 (ii) 16 : 15 (iii) 25 : 21 (iv) 13 : 6 (v) 4 : 1
(vi) 1 : 50 (vii) 3 : 4 (viii) 3 : 7 (ix) 6 : 5

3- (i) 14 : 9 (ii) 3 : 25 (iii) 9 : 28

4- (i) 15 : 23 (ii) 23 : 17 (iii) 23 : 12 (iv) 12 : 17 (v) 4 : 5 (vi) 17 : 15

Exercise 1.4

1- 1 : 1 2- 16 : 15 3- 3 : 4 4- (i) 1 : 3 (ii) 1 : 3 (iii) 1 : 9 5- 2 : 1

6- (i) 3 : 4 (ii) 2 : 3 (iii) 1 : 2 7- 3 : 7 8- (i) 8 : 7 (ii) 7 : 1 (iii) 8 : 1

9- (i) 3 : 4 (ii) 9 : 16 10- (i) 2 : 1 (ii) 1 : 3

Exercise 1.5

1- 20 2- 9 suits 3- 5 liters 4- 75 km / hr

5- 8 days 6- 120 bicycles 7- 864 fans 8- 720 soaps

9- Rs.2240 10- Rs.18200 11- 110 cows 12- 2700 bottles

Review Exercise -1

1- Encircle the Correct Answer.

(i) b (ii) b (iii) a (iv) d (v) c (vi) b (vii) c (viii) a (ix) a (x) b

2- Fill in the blanks.

(i) 450 (ii) $\frac{3}{20}$ (iii) 28 % (iv) 67% (v) 29 % (vi) antecedent

(vii) consquent (viii) extremes (ix) means (x) 10 : 9

3- 30% 4- Rs.7000 5- (i) 2:1 (ii) 22 : 5 (iii) 11:3

6- 2 : 3 7- 16 days 8- Rs.9720

Exercise 2.1

1- Rs.27750 2- Rs.7500 3- Rs.15,000 4- Rs.36,250 5- Rs.60,000

6- Rs.17500 7- widow Rs.93750; son Rs.262500; daughter Rs.131250

8- widow Rs.50,000; daughter Rs.87500 9- Rs.3,75,000

10- widow Rs.2,50,000; son Rs.8,75,000

11- widow Rs.60,000; son Rs.84,000, daughter Rs.42000 12- 100,000

Review Exercise 2

1- Encircle the Correct Answer.

(i) a (ii) c (iii) a (iv) a (v) c (vi) b (vii) a (viii) c (ix) a (x) a

2- Fill in the blanks.

(i) 2.5 % (ii) 10 % (iii) 5 % (iv) Rs.5000 (v) Rs.10,000

(vi) $\frac{1}{8}$ (vii) $\frac{1}{4}$ (viii) $\frac{1}{2}$ (ix) 2:1 (x) $\frac{2}{3}$

3- Rs.37500 4- Rs.24500 5- widow : Rs.5,62,500, Sons : Rs.19,68,750

6- widow : Rs.6,00,000, Son : Rs.8,40,000 each, daughter Rs.4,20,000 each.

Exercise 3.1

1- (i) Rs.1045 (ii) Rs.1463 (iii) Rs.10560 (iv) Rs.119700 (v) Rs.494.40 (vi) Rs.729.60

2- Rs.2,73,000 3- (i) Rs.640 (ii) Rs.925 (iii) Rs.1560 (iv) Rs.3000 (v) Rs.75

4- $12\frac{1}{2}\%$ 5- 25 % 6- Rs.1260 7- Rs.213 8- Rs.1215

Exercise 3.2

1- (i) Rs.684.32 (ii) Rs.2622 (iii) Rs.364.08 2- (i) Rs.560 (ii) Rs.975 (iii) Rs.2500

3- 5 % 4- 7.5 % 5- Rs.400 6- Rs.920 7- Rs.11730.60 8- 20 %

Exercise 3.3

1- 60,000; 40,000; 1,00,000

2- Ali's share Rs.22,500; Daniyal's share Rs.33,750; Abdullah's share Rs.56,250

3- 180; 240; 300

4- Profit 1st partner Rs.1,00,000; 2nd partner Rs.1,50,000; 3rd partner Rs.3,50,000
Amount 1st partner Rs.600,000. 2nd partner Rs.900,000. 3rd partner Rs.2100,000

5- 39200 6- (a) Rs.70 Total amount Rs.154

Review Exercise 3

1- **Encircle the Correct Answer.**

(i) c (ii) b (iii) b (iv) b (v) a

2- **Fill in the blanks.**

(i) cost price (ii) sale price (iii) profit (iv) $\frac{loss}{CP} \times 100$ (v) 100 − discount%

3- 5.5 % 4- 16.6 % 5- 9000; 15000; 9000 6- (i) 60000 (ii) 36000

Exercise 4.1

1- 161.406 2- Rs.111724.5 3- Loss Rs.15

4- (i) Rs.12615 (ii) Rs.12517.42 (iii) Rs.97.58 5- (i) Rs.4.63 (ii) 1.89375 Riyals

6- 15772 7- Profit = Rs.1099.58 8- Profit = Rs.4759.5

9- T.T buying is applicable for Rs.10055.20 (ii) Rs.1857.37

Exercise 4.2

1- Rs.1100 2- Rs.132 3- 10 years 4- 4.25% per year 5- Rs.1019.50 6- $7\frac{1}{2}$%P.A

7- (a) Rs.6720, Rs.18720 (b) 4years, Rs.720 (c) Rs.300, Rs.408 (d) 4%, Rs.4200 (e) Rs.3600, 5%
(f) 7%, Rs.1989 (g) 6%, Rs.540 (h) Rs.1200, $1\frac{1}{2}$ years 8- Rs.16 9- Rs.6180 10- Rs.20,000

11- (i) Rs.94.50 (ii) Rs.257.34 (iii) Rs.1244.03 (iv) Rs.149.4 (v) Rs.2422.94

12- Rs.5829.57 13- Rs.903.13 14- 9550.87

Exercise 4.3

1- Rs.1520.50 2- Rs.88.50 3- Rs.65000 4- Rs.7084 5- Rs.3600
6- Rs.35000 7- Rs.157.49 8- Rs.3040 9- Rs.6480

Exercise 4.4

1-

Amount of policy	Yearly premium	Half yearly premium	Quarterly premium	Monthly premium
(i) 50,000	2250 + 125	1235	641.50	213.75
(ii) 100,000	4500 + 200	2444	1269.00	423.00
(iii) 150,000	6750 + 200	3614	1876.50	625.30
(iv) 200,000	9000 + 200	4784	2484.00	828.00

2- Rs.(50,000 + 1,20,000 + 4500) = Rs.1,74,500 3- Rs.(150000 + 135000 + 121500 + 0) = Rs.406500
4- Rs.700,000, Rs.159050 5- Rs.4,32,250 6- Rs.(36,125 + 32512.50) = Rs.68,637.50
7- Rs.(26,250 + 23,625 + 21,262.50 + 19136.25) = 90273.75 benefit Rs.9726.25
8- Rs.800,000 (approx.), Rs.23400, Zero.

Exercise 4.5

1- (a) i. Rs.90 ii. 25% (b) i. Rs.150 ii. $16\frac{2}{3}\%$ (c) i. Rs.3000 ii. 12% 2- (i) Rs.21 (ii) Rs.288
3- (a) i. Rs.236 ii. 18% (b) i. Rs.517.50 ii. 15% (c) i. Rs.1960 ii. 22.5%
4- (a) i. Rs.63 ii. 7% (b) i. Rs.200 ii. 15% (c) i. Rs.75 ii. $16\frac{2}{3}\%$
5- (i) Rs.3716.80 (ii) Rs.516.80 (iii) 16.15%

Review Exercise 4

1- Encircle the Correct Answer.

(i) a (ii) b (iii) a (iv) c (v) a (vi) a (vii) a (viii) c (ix) a

2- Fill in the blanks.

(i) cheque (ii) pay order (iii) ATM (iv) profit (v) rate
(vi) time (vii) mark-up (viii) premium (ix) insurer (x) insured

3- Rs.515,850 4- Rs.700,000, Rs.159,050 5- Markup Rs.2,770 Prinicipal Rs.27,230

Exercise 5.1

1- Rs.28000 2- Rs.6400, Rs.46400 3- Rs.18,00,000, Rs.28,00,000
4- Rs.2,40,000, Rs.3,20,000 5- Rs.11250, Rs.13950 6- Rs.16350 7- Rs.21375
8- Rs.6875

Exercise 5.2

1- (i) Rs.5124.29 (ii) Rs.9168.056 (iii) Rs.3622.84 (iv) Rs.950.12
(v) Rs.14771.5 (vi) Rs.9848.75

2- (i) Rs.1155.39 (ii) Rs.718.09 (iii) Rs.1540 (iv) Rs.1021.41

3- (i) Rs.3153.50 (ii) Rs.3808.00 (iii) Rs.4462.50 (iv) Rs.1606.50

(v) Rs.2856 (vi) Rs.1874.25

Exercise 5.3

1- 48 hrs, Rs.1920 2- Rs.4800 3- Rs.17640 4- Rs.26500 5- Rs.41,500

6- (i) Rs.1975 (ii) Rs.2725 (iii) Rs.2875 7- Rs.10,800

Review Exercise 5

1- Encircle the Correct Answer.

(i) a (ii) b (iii) a (iv) b (v) a (vi) a (vii) c

2- Fill in the blanks.

(i) tax (ii) direct tax (iii) indirect tax (iv) sales tax (v) exercise duty

(vi) property tax (vii) income tax (viii) Rs.90,000 (ix) Rs.1200 (x) 4,50,000

3- Rs.19200 4- Rs.24750 5-Rs.15867.37

6- (i) Rs.1017.48 (ii) Rs.736.02 (iii) Rs.989.23 (iv) Rs.1055 7- Rs.70500

Exercise 6.1

1- (i) Redical $\sqrt{3}$; Redicand 3 (ii) Redical $\sqrt{a}$; Redicanda (iii) Redical $\sqrt{11}$; Redicand11
(iv) Redical $\sqrt{6}$: Redicand6 (v) Redical $\sqrt{5}$: Redicand5 (vi) Redical $\sqrt{13}$; Redicand13

2- (i) $a^{3/2}$ (ii) $a^{3/5}$ (iii) $a^{-k/p}$ (iv) $a^{-k/b}$

3- (i) $\sqrt{25}, 5$ (ii) $\sqrt[3]{64}, 4$ (iii) $\sqrt[4]{81}, 3$ (iv) $\sqrt[3]{27}, 3$ (v) $\sqrt[3]{(27)^2}, 9$ (vi) $\sqrt[3]{\frac{1}{8}}, \frac{1}{2}$

(vii) $\sqrt[3]{(1000)^2}; 100$ (viii) $\sqrt{64}; 8$

4- (i) a^8 (ii) a^5 (iii) $3a^3$ (iv) $2a^3$ (v) x^8 (vi) $3x^5$ (vii) $5x^3y^5$

(viii) $(8+y)^{7/2}$ (ix) $2x^{1/2}y^{3/2}$ (x) $\frac{x^{5/4}y^{3/2}}{z^{1/2}}$ (xi) $\frac{2x^{1/3}}{(x+y)^{1/3}}$ (xii) $\frac{y^{n/p}}{a^{m/p}}$

5- (i) $\sqrt{21}$ (ii) $\sqrt[5]{512}$ (iii) $\sqrt[5]{2187}$ (iv) $\frac{1}{\sqrt[18]{2}}$ (v) $\sqrt[5]{59}$ (vi) $\sqrt{\frac{1}{3}}$ (vii) $\sqrt[12]{a^{11}}$

(viii) $\sqrt[4]{x^{24/7}y}$ (ix) $y\sqrt{x^9}$ (x) $\sqrt[6]{x}$ (xi) $\sqrt[4]{x^4y^5}$ (xii) $\sqrt[24]{a^{11}b^{26}}$ (xiii) $x^{29/60}y^{11/10}$

Exercise 6.2

1-

Q.No.	(i)	(ii)	(iii)	(iv)	(v)	(vi)
Base	x	x	$4y$	$x-2$	x	x
Exponent	3	9	3	3	5	2

2- a^6b^9 3- $\frac{x^{4/3}}{y}$ 4- $\frac{1}{y^{2a+3b}}$ 5- $\frac{1}{2x^{1/2}y^{1/2}}$ 6- $\frac{4}{9a^2x^2}$ 7- $16ac^4$

8- $\frac{3}{ba^{1/2}}$ 9- $\frac{a^{11/2}c^{3/2}}{b^3}$ 10- b^3 11- $\frac{x^{11}y^6}{16}$ 12- ab^{13} 13- a^4b^3c

14- $\frac{4b}{9ac^9}$ 15- $2^{16}\times 3^8$ 16- $2^{14}\times 3^7$ 17- $\frac{1}{2a^4b^6}$ 18- 2187

19- $\frac{3^7}{2^6}$ 20- $\frac{2^3}{3^6}$ 21- $\frac{125}{2187}$ 22- $a^{7/6}b^{11/12}$ 23- $a^{5/6}b^{3/2}$ 24- $a^3b^2c^{3/2}$

25- $a^{1/2}b^{23/72}$ 26- $a^{11/12}$

27- (i) $4^{4/5}$ (ii) $2^{1/2}$ (iii) $10x^{8/15}$ (iv) $x^{11/20}$ (v) $2y^{5/7}$ (vi) $5x^2$

28- (i) $ab^{3/2}$ (ii) $x^{4/5}y^{5/9}$ (iii) $6a^{8/5}b^{17/15}$ (iv) $2x^{19/28}y^{2/5}$ (v) $x^{19/6}y^{5/6}z^{5/6}$

29- (i) $3^{1/6}$ (ii) $x^{4/3}$ (iii) $\frac{1}{2}x^{3/20}$ (iv) $\frac{5}{4}y^{7/20}$ (v) $x^{5/3}y^{7/5}$ (vi) $a^{7/45}b^{4/15}$

(vii) $2x^{2/15}y^{3/4}$ (viii) $\frac{1}{4}a^{11/20}b^{7/20}$

Exercise 6.3

1- 5.1×10^{-2} 2- 8.899×10^1 3- 4.24×10^{-1} 4- 2.566324×10^6

5- 7.5×10^{-7} 6- 8600 7- 0.00001345 8- 0.0000000051

9- 0.0000000525 10- 0.0006365 11- 4.8×10^2 12- 1.044×10^6

13- 3.126×10^4 14- $4.5\times 10^3\,m$ 15- 6.4×10^6

Exercise 6.4

1- (i) 3 (ii) 3 (iii) 0 (iv) $\bar{1}$ (v) $\bar{2}$ (vi) $\bar{4}$

2- (i) 1.7170 (ii) 0.7996 (iii) $\bar{1}.7873$ (iv) $\bar{3}.7559$ (v) $\bar{5}.4771$

3- (i) 0.8044 (ii) $\bar{1}.8044$ (iii) $\bar{3}.8044$ **4-** (i) 0.01090 (ii) 1.444 (iii) 26530

Exercise 6.5

1- (i) 2 (ii) 2 (iii) $\frac{5}{2}$ **2-** (i) 1 (ii) 3 (iii) 4 (iv) 2 (v) 3

3- (i) $\log \frac{1.3472 \times 22.79}{5}$ (ii) $\log \frac{22.13 \times 0.354 \times 7}{3}$ (iii) $\log \frac{57.86 \times 4.385}{2.391 \times 3.072}$

4- (i) 1.923 (ii) 19.19 (iii) 0.9945

8- (i) 0.3291 (ii) 19.19 (iii) 14.139 (iv) 0.3466 (v) 160.4 (vi) 48.01

(vii) 1964 (viii) 2.082 **9-** 27.71 **10-** 67.39 **11-** 0.1224

Review Exercise 6

1- Encircle the Correct Answer.

(i) b (ii) a (iii) b (iv) c (v) a (vi) b (vii) a (viii) b (ix) d (x) a

2- Fill in the blanks.

(i) radical (ii) radicand (iii) exponent (iv) base (v) common logarithm
(vi) characteristic (vii) mantissa

3- (i) $x^{3/2}y^{-5/6}$ (ii) $a^{7/8}b^{7/12}$ **4-** (i) $x^{1/3}y^{19/24}$ (ii) $\frac{1}{40}$ **6-** (i) 0.2983 (ii) 5.158 (iii) 0.2465

Exercise 7.1

1- (i) 4,5,6 (ii) $-1, 8, -27$ (iii) 8,11,14 (iv) $\frac{2}{7}, \frac{1}{3}, \frac{4}{11}$ (v) $1, \frac{1}{9}, \frac{1}{25}$ (vi) 4,5,6

(vii) $\frac{1}{3}, \frac{1}{9}, \frac{1}{27}$ (viii) $-2, 1, 4$ (ix) 1,3,12

2- (i) 51 (ii) 20160 (iii) $\frac{1}{243}$ (iv) 15 (v) $\frac{5}{11}$ (vi) 17

3- (i) 34,42,51,61 (ii) 63,127,255,511 (iii) 112,288,704,1664

(iv) 25,27,30,33 (v) 20,24,28,32 (vi) 12,14,16,18

Exercise 7.2

1- (i) 243 (ii) -58 (iii) -82 (iv) 74 (v) -74

2- (i) 16 (ii) 132 (iii) 4 (iv) 65 (v) 79 (vi) 193 **3-** 12 **4-** 39

5- 42 **6-** 20 **7-** $12-11x$ **8-** $3n+24$ **9-** $\left(\frac{3}{3n+1}\right)^2$ **10-** $-2,1,4,7,10,13,\ldots\ldots$

Exercise 7.3

1- (i) 2 (ii) $x+3$ (iii) $2\sqrt{7}$ (iv) x^2+1 **2-** 0,9 **3-** 13,15,17

4- $\frac{9\sqrt{2}}{4},\frac{7\sqrt{2}}{2},\frac{19\sqrt{2}}{4}$ **5-** $\frac{38}{7},\frac{41}{7},\frac{44}{7},\frac{47}{7},\frac{50}{7},\frac{53}{7}$ **6-** $\frac{17}{2},9,\frac{19}{2},10,\frac{21}{2},11,\frac{23}{2}$

7- 15 **8-** 70 **9-** 1,17

Exercise 7.4

1- 8192 **2-** 118098 **3-** 48 **4-** 324 **5-** (i) 3 (ii) 5 (iii) 10

6- 576 **7-** 18225 **9-** $\left(\pm\frac{2}{3}\right)^n$ **10-** 2,6,18 or 18,6,2

11- $\frac{1}{x^{28}}$ **12-** x^{2p-1}

Exercise 7.5

1- (i) $\pm 3\sqrt{5}$ (ii) ± 6 (iii) $\pm 2\sqrt{2}$ **2-** (i) 2,4 (ii) 9,27 **3-** (i) 2,4,8 (ii) 2,8,16

4- 6,12,24,48 **5-** 2,8;8,2 **6-** 2,18 **8-** 80,40,20,10

9- 162,54,18 **10-** $\frac{1}{2},2,8,32$ **11-** $-28,14,-7,\frac{7}{2},\frac{-7}{4},\frac{7}{8}$ **12-** $\frac{16}{27},\frac{8}{9},\frac{4}{3},2,3$

Review Exercise 7

1- Encircle the Correct Answer.

(i) b (ii) c (iii) a (iv) a (v) c (vi) b (vii) a (viii) c (ix) a (x) b

2- Fill in the blanks.

(i) a_n (ii) 15 (iii) nth term (vi) 10 (v) $\dfrac{a+b}{2}$

(vi) common ratio (vii) ar^{n-1} (viii) $\pm\sqrt{ab}$ (ix) 4 (x) 3n + 24

3- $a_n = 2n+1,\ 37$ 4- $a_n = \left(\dfrac{3}{2n+3}\right)^3$ 5- 32 6- $(3)^{11}$ 7- 1,2,4,8

8- 2,6,18, or 18,6,2

Exercise 8.1

1. (i) {1,4,6,7,8,9} (ii) {3,4,5,6,7,8,9} (iii) {4,7} (iv) {4}
   (v) {1,3,4,5,6,7,8,9} (vi) {4}

8. (i) A (ii) A (iii) A (iv) Φ (v) Φ (vi) $A' \cup B'$
   (vii) $A' \cap B'$ (viii) A (ix) Φ (x) Φ

Exercise 8.2

1. {(3,1), (3,3), (5,1) (5,3), (6,1), (6,3)} , {(1,3), (1,5) (1,6), (3,3), (3,5), (3,6)}
   Dom $A \times B$ = {3,5,6} Range $A \times B$ = {1,3}
   Dom $B \times A$ = {1,3} Range $B \times A$ = {3,5,6}

2. {(−2,−2), (−2,1)}, {(−2,4) (1,−2)} Dom = {−2} Dom = {−2,1}
   Range = {−2,1} Range = {4,−2}

3. (i) 2^9 (ii) 2^{12} 4. {(2,2), (3,3), (3,2)}

Review Exercise 8

1- Encircle the Correct Answer.

(i) b (ii) b (iii) a (iv) b (v) a (vi) a (vii) b (viii) b (ix) c (x) d

2- Fill in the blanks.

(i) $A' \cap B'$ (ii) $A' \cup B'$ (iii) $(A \cup B) \cup C$ (iv) $(A \cap B) \cap C$

(v) commutative law as under union (vi) commutative law under intersection

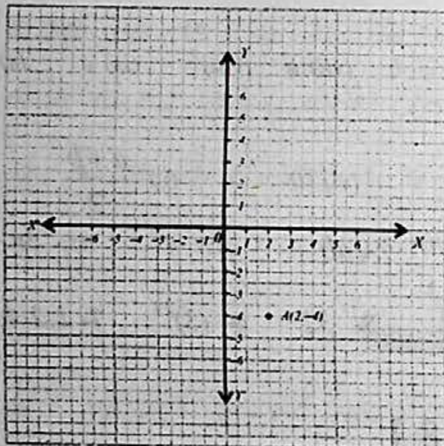
(vii) binary relation (viii) {1,3,5} (ix) {2,4,6} (x) set B

6- (i) 2^{16} (ii) 2^6 7- Dom (R) = {2,4} Range (R) = {2,5}

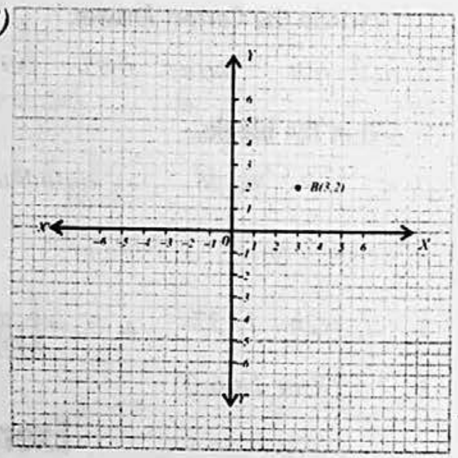
Exercise 9.1

1-

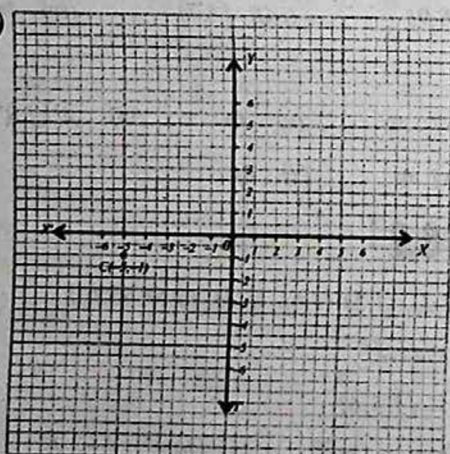
(i)



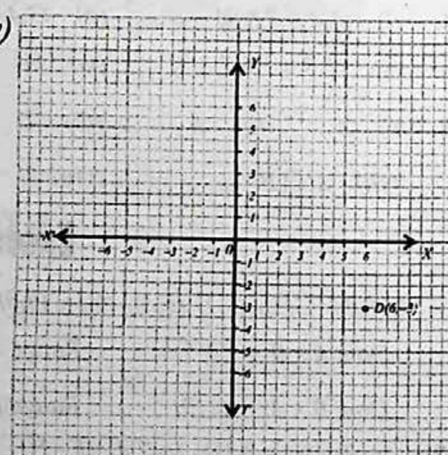
(ii)



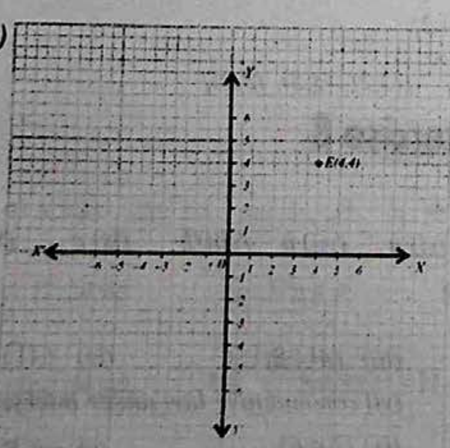
(iii)



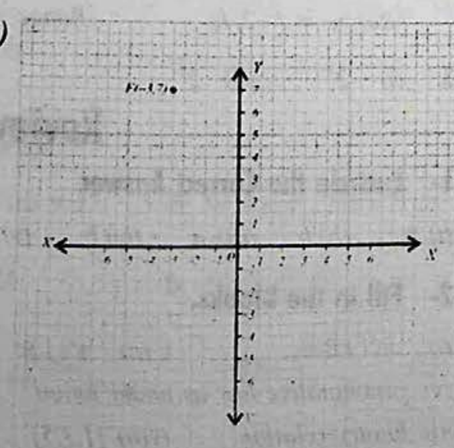
(iv)



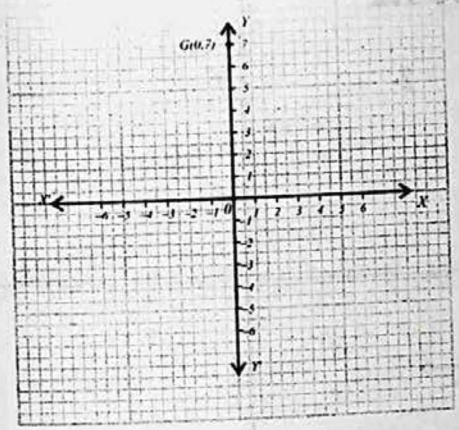
(v)



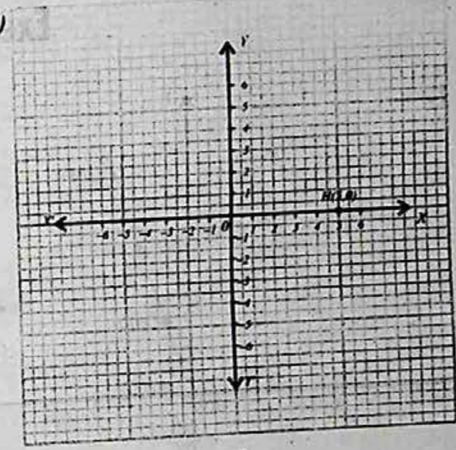
(vi)



(vii)

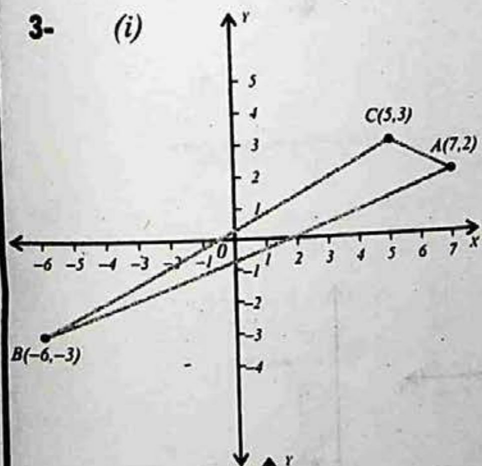


(viii)

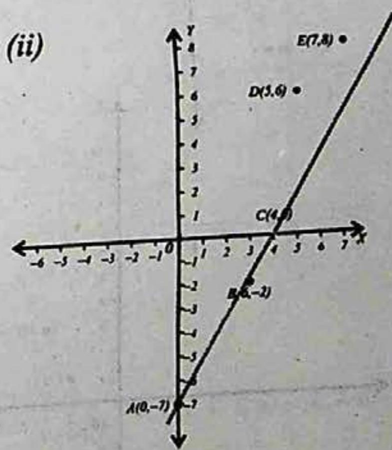


2- (i) (0,0) (ii) (−5,0) (iii) (3,0) (iv) (0,4) (v) (0,−6)

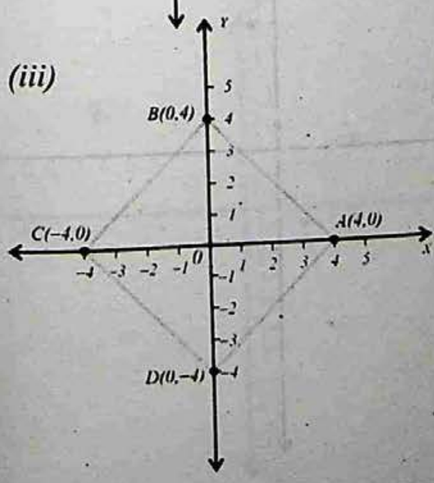
3- (i)



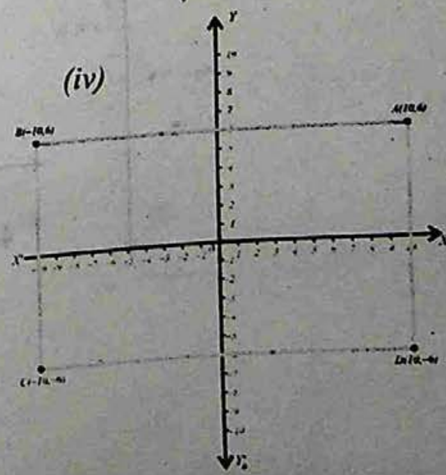
(ii)



(iii)

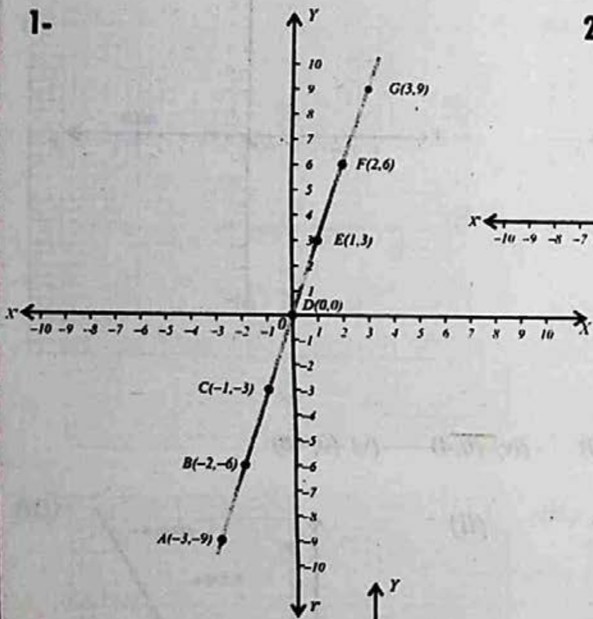


(iv)

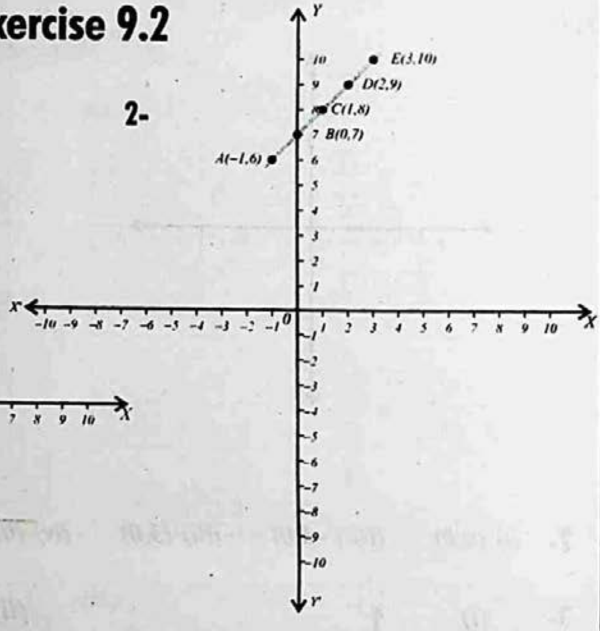


Exercise 9.2

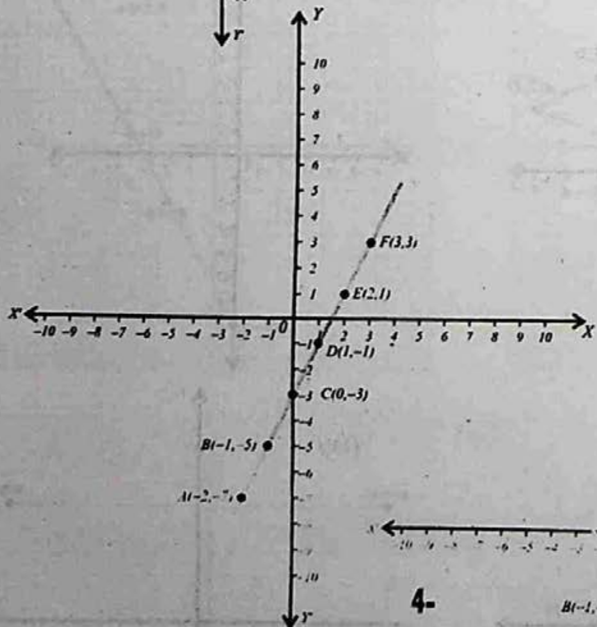
1-



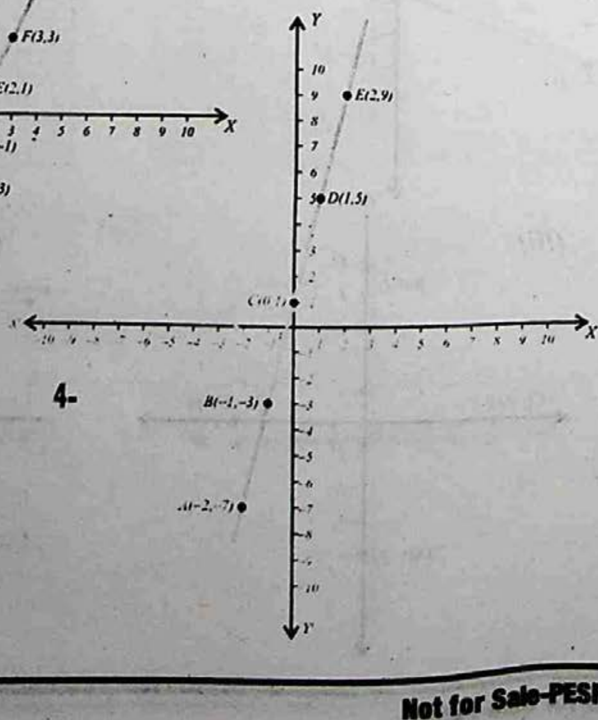
2-



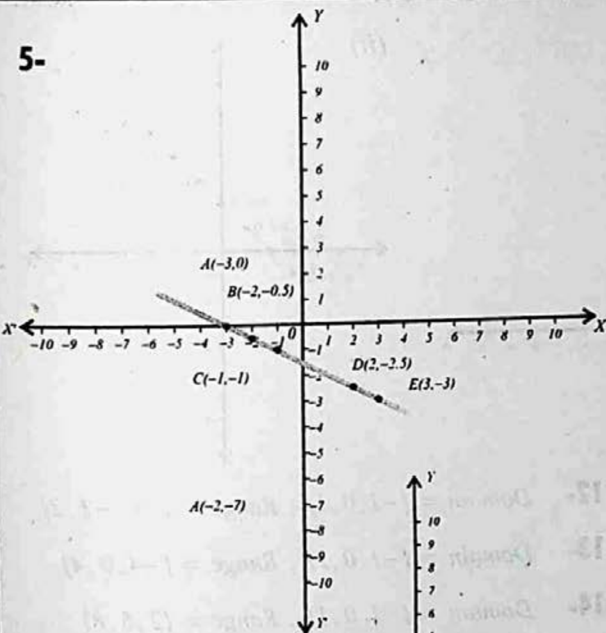
3-



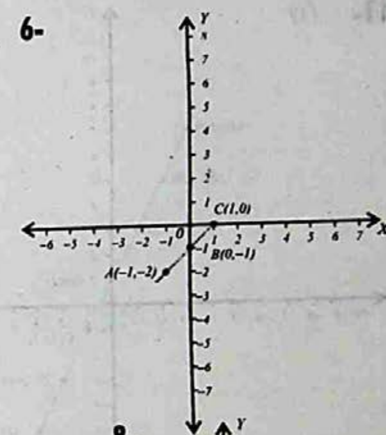
4-



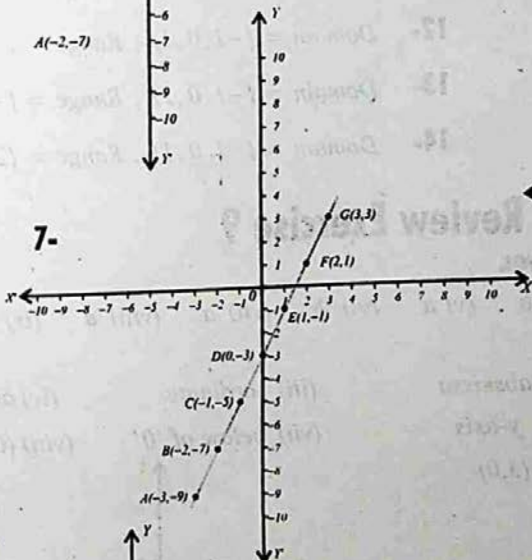
5-


A(−3,0)
B(−2,−0.5)
C(−1,−1)
D(2,−2.5)
E(3,−3)
A(−2,−7)

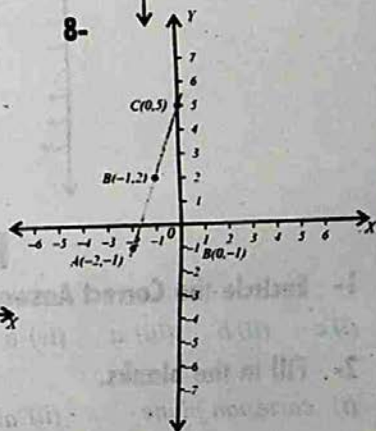
6-


C(1,0)
B(0,−1)
A(−1,−2)

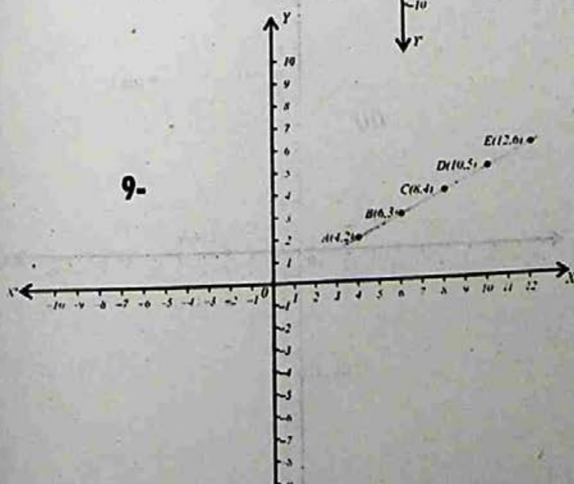
7-


G(3,3)
F(2,1)
E(1,−1)
D(0,−3)
C(−1,−5)
B(−2,−7)
A(−3,−9)

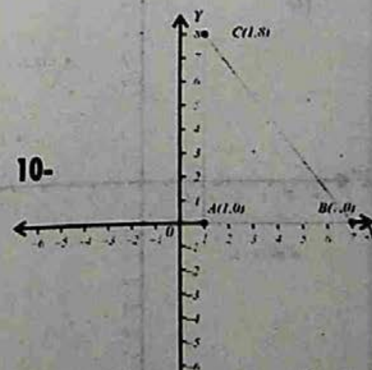
8-


C(0,5)
B(−1,2)
A(−2,−1)
B(0,−1)

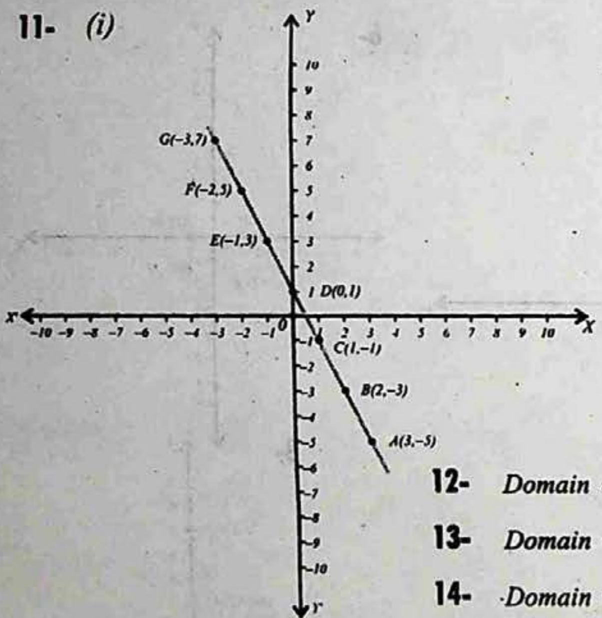
9-



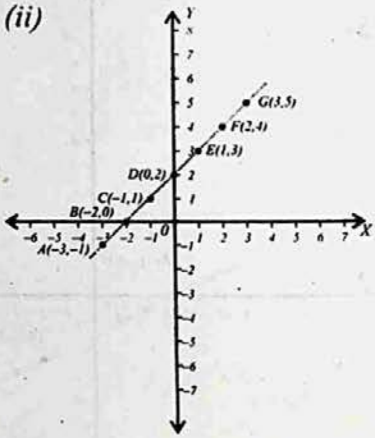
10-



11- (i)



(ii)



12- Domain $= \{-1, 0, 1\}$, Range $= \{-4, -1, 2\}$

13- Domain $= \{-1, 0, 1\}$, Range $= \{-4, 0, 4\}$

14- Domain $= \{-1, 0, 1\}$, Range $= \{2, 5, 8\}$

Review Exercise 9

1- Encircle the Correct Answer.

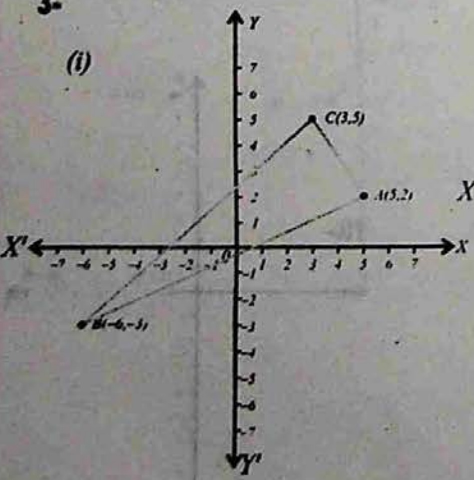
(i) c (ii) b (iii) a (iv) b (v) a (vi) b (vii) a (viii) a (ix) b

2- Fill in the blanks.

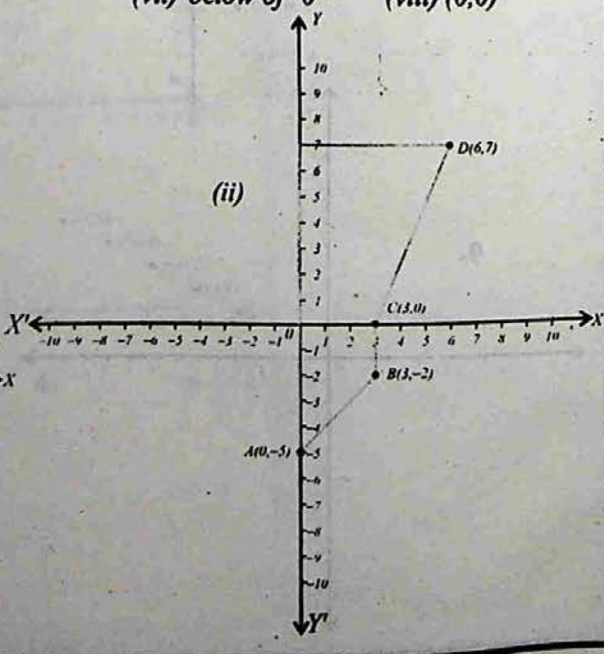
(i) cartesian plane (ii) abscissa (iii) ordinate (iv) ordered pair
(v) x-axis (vi) y-axis (vii) below of '0' (viii) (0,0)
(ix) $y = c$ (x) (3,0)

3-

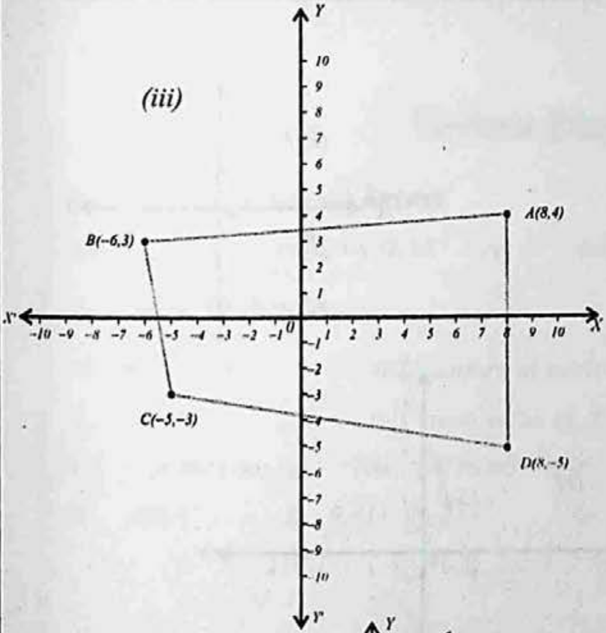
(i)



(ii)

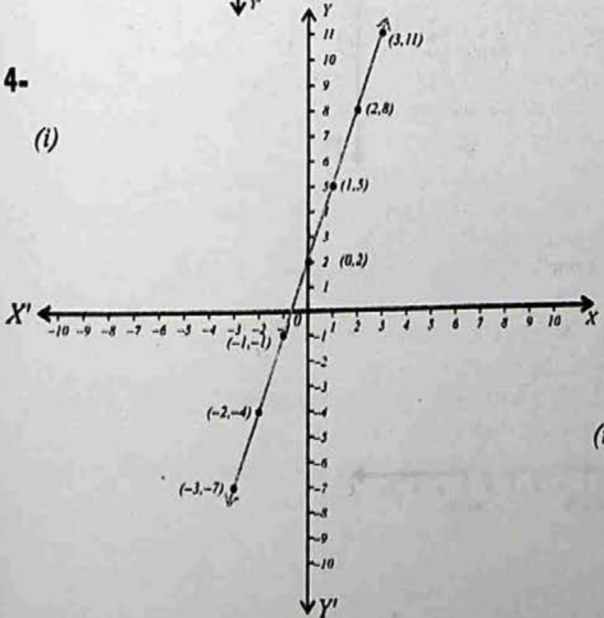


(iii)

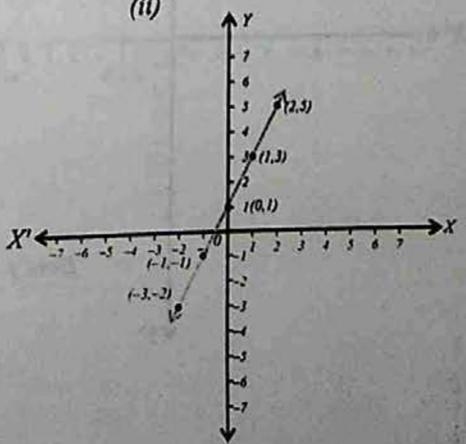


4-

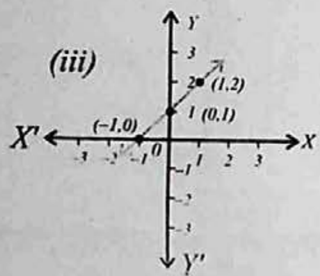
(i)



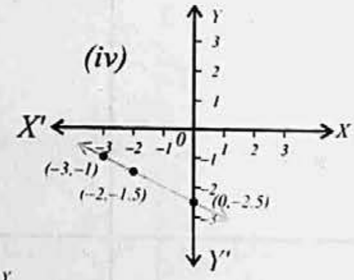
(ii)



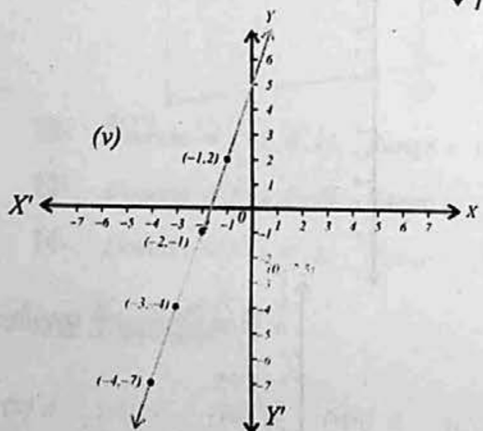
(iii)



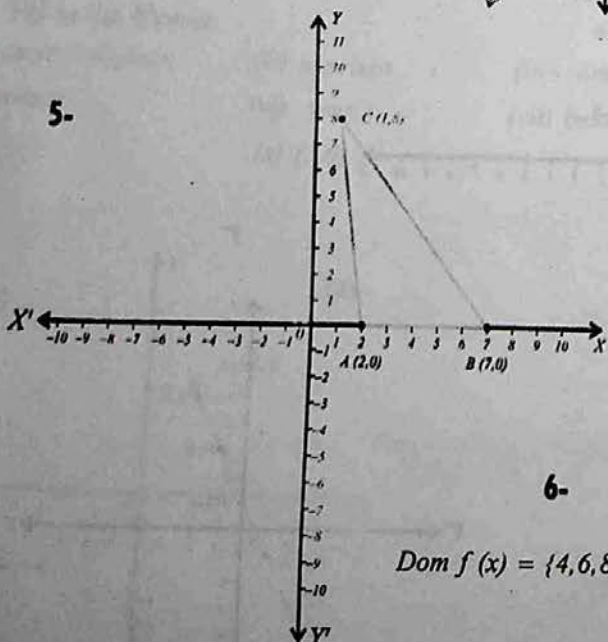
(iv)



(v)



5-



6-

Dom $f(x) = \{4, 6, 8, 10, 12\}$, *Range* $f(x) = \{2, 3, 4, 5, 6\}$

Review Exercise 10

1- Encircle the Correct Answer.

(i) c (ii) d (iii) a (iv) b (v) a (vi) a (vii) a (viii) b (ix) a (x) a

2- Fill in the blanks.

(i) histogram (ii) measure of central tendency (iii) mean (iv) medain

(v) mode (vi) mean value of $x_1, x_2, \ldots\ldots\ldots, x_n$

(vii) geometric mean (viii) harmonic mean (ix) weighted mean (x) arithmetic mean

3- *3.286* **4-** *9.811; 96.255* **5-** *7; 8; 8* **6-** *6.875* **7-** *4,7*

GLOSSARY

Unit-1 PERCENTAGE, RATIO AND PROPORTION

Percentage: Percentage means out of hundred.

Ratio: A comparison between the two like quantities is called a ratio.

Antecedent: In a ratio $a:b$, "a" is called the antecedent.

Consequent: In a ratio $a:b$, "b" is called the consequent.

Proportion: The equality of two ratios is called proportion.

Extremes and Means: If $a:b::c:d$, then "a" and "d" the first and fourth terms are called extremes and "b" and "c" the *2nd* and *3rd* terms are called means.

Direct Proportion: The relation between two ratios in which an increase in one quantity causes a proportional increase in the other quantity or decreases in one quantity causes a decrease in another quantity is called direct proportion.

Inverse Proportion: The relationship between two ratios in which increase in one quantity causes a proportional decreases in the other quantity or vice versa is called inverse proportion.

Compound Proportion: The relationship between two or more proportions is called a compound proportion.

Unit-2 ZAKAT, USHR AND INHERITANCE

Zakat: Zakat is a transfer payment which Sahib-e-Nisab Muslims pay at given rates by them according to sharia or through the Islamic rate to the poor and needy in or after month of Rajab.

Sahib-e-Nisab Muslim: A Muslim who owns or keeps in his / her possession at least *7.5* tola gold or *52.5* tola silver or cash money equivalent to value for one year is considered a sahib-e-nisab Muslim.

Exposed wealth: This includes agriculture goods camels, sheep, goats, minerals, business inventories etc.

Unexposed wealth: This includes gold, silver, cash money, liquid assets etc.

Rate of Zakat: Rate of zakat is *2.5 %* of the total value of the goods or money.

Ushr: It is a tax paid at the rate of *10 %* from agriculture produce of land which is irrigated by natural resources and *5 %* by artificial.

Inheritance: Distribution of remaining inheret amongst the heirs according to sharia after the burial of a deceased.

Unit-3 BUSINESS MATHEMATICS

Cost Price: The price at which a particular item is purchased is called cost price. It is denoted by "*CP*".

Sale Price: The price at which an article is sold out is called the sale price. It is denoted by "*SP*".

Profit: If the selling price of an article is greater than its cost price, then the difference of these two is the profit earned. It is denoted by "*P*".

Loss: If the selling price of an article is less than its cost price, then the difference of these two is the loss. It is denoted by "*L*".

Discount: Some times a rebate is declared on the selling price of an article, this rebate is called the discount.

Marked Price (MP): The price tagged on a card of each and every article in a shop is known as the marked price, It is denoted by "MP".

Unit-4 FINANCIAL MATHEMATICS

Current Account: A running account which continuously remains in operation due to its liquidity.

Saving Account: It is meant to encourage thrift and promote saving among the persons of small means. The bank pays nominal interest half yearly on the basis of monthly balance to the depositors.

PLS Saving Account: Profit and loss sharing account opened with small amount with profit earned on loss sustained at the end of each half year / full year depending upon the mode of payment.

Fixed / Time deposit Account: The deposits kept with the bank in a account for a certain period of time ranging from 3 months to 5 years.

Foreign Currency Accounts: Account maintained with the bank in foreign currency like dollars, pounds and Euro etc.

Negotiable Instruments: It means a promissory note, a bill exchange or cheque payable whether to be order or bearer of the instruments.

Insured: The person or entity whose insurance is being done is called "the insured".

Insurer: The company under taker the act of insurance is called the insurer.

Insured or Insurant: A person to whom an insurance policy issued, the beneficiary in a contract issuance is called insured or insurant.

Insurance Policy: The contract which is executed between two parties is called insurance policy.

Premium: The periodic installment to be paid by the insured is called premium.

Maturity: The time-period agreed upon by both the parties (insured and insurer is called maturity.)

Bonus: The agreed amount to be paid back on maturity or expiry of the agreed period, includes the actual amount paid in installments plus profit is termed as bonus.

Cheque: A bill of exchange drawn on a specific banker and not expressed to be payable other wise on demand.

Pay Order: A cheque like instruerent issued by bank on the request of its customers.

Bank Draft: An order to pay money, drawn by one office of a bank upon another office of the same bank for a sum of money payable to order on demand.

On-Line Banking: This system indicates that a direct connection is made to centralize computer system for authorization or validation before a transaction is executed.

ATM Card: It is a payment card issued to a person for activating automated teller machine computer based terminal which allow consumers to make deposits and with draws.

Credit Card: A card indicating that the holder has been granted a line of credit enabling the holder to make purchases and or with draw cash.

ATM: A machine installed by the bank to dispense cash to its account holders.

Profit: Profit is the amount which is paid by the bank on the deposits maintained by the client with the bank.

Principal: The amount / capital borrowed or lent is called principal.

Rate: The percentage of interest charged is called rate.

Time: The period of the loan or deposit is called the time.

Amount: When the interest is added to the principal, the sum is called the amount.

Mark Up: The interest earned by the bank is named as mark up.

Leasing: Lease is a contract where by the owner of an asset, the lessor, gives the hirer, the lessee, the right to use the asset for a specific period in exchange of rental payment.

Down Payment: The customer is required to deposit the payment with the bank along with the application form.

Unit-5 CONSUMER MATHEMATICS

Tax: Money that must be paid to the state, charged as a proportion of income and profits or added to the cost of same goods and services.

Direct Tax: These are the taxes which are charged on income, property and profits in the from of income tax, property tax.

Indirect Tax: Indirect taxes include duties, motor vehicle taxes, goods and services taxes *(GST)* general sale tax and value added taxes etc.

Sales Tax: When we buy article we have to pay a certain amount of tax as the value added tax in addition to the price of the article. This tax usually given as a certain percentage of the selling price. In Pakistan sales tax of *17%* is imposed on goods bought and services rendered.

Excise Duty: It is the form of a tax which the buyer pay on a manufactured item at the time of purchase.

Property Tax: A property tax is changed on the owner of land, house, flats or building at a standard rate of *16%* on annual value of the property.

Income Tax: It is the tax charged on all taxable incomes during the year from *1st* July to next *30th* June.

Unit-6 EXPONENTS AND LOGARITHMS

Rational Number: A number of the form $\frac{p}{q}, q \neq 0, p, q \in I$ is called a rational number.

Irrational Number: A number of form $\sqrt{2}, \sqrt{3}, \sqrt{5}, \pi$... are called irrational numbers.

Radicals: If $\sqrt[n]{a}$ is irrational, where "a" is rational number, then $\sqrt[n]{a}$ is called a radical.

Radicands: The symbol $\sqrt[n]{\ }$ is called radicands .

Base and Exponent: For any real number "a" and a positive integer "n" we define $a^n = a \times a \times a \times .... \times a$ *(n times)*. Here "a" is called the base and "n" the exponent.

Pure Radical: A radical which has unity (one) only as rational factor, the other factor being irrational is called a pure radical.

Mixed Radical: A radical which has a rational factor other than unity, the other factor being irrational is called a mixed radical.

Common Logarithm: The logarithm calculated to the base *10* is called a common logarithm.

Characteristic and Mantissa: The logarithm of a number consists of two parts, the integral part is called the characteristic and the decimal part is called the mantissa.

Unit-7 ARITHMETIC AND GEOMETRIC SEQUENCES

Sequence: A sequence is a function whose domain D is a set of positive integers.

Arithmetic Sequence: A sequence in which each term is obtained from the previous term by adding a fixed number is called an arithmetic sequence.

Arithmetic Mean: A number "A" is said to be an arithmetic mean between the two numbers a and b if a, A, b is arithmetic sequence.

Geometric Sequence: A sequence in which each term is obtained from the previous term by multiplying with a common ratio is called a geometric sequence.

Geometric Mean: A number "G" is said to be a geometric mean between the two numbers a and b if a, G, b is a geometric sequence.

Unit-8 SETS AND FUNCTIONS

Set: A collection of well defined distinct objects is called a set.

$N = \{1, 2, 3, ...\}$ is called set of natural numbers.

$M = \{0, 1, 2, 3, ...\}$ is called set of whole numbers.

$I = \{...... -1, 0, 1, 2, 3, ...\}$ is called set of integers.

$Q = \{p/q\ , q \neq 0,\ p, q \in I\}$ is called set of rational numbers.

$Q' = A$ set of irrational numbers.

$R = QUQ' =$ A set of real numbers.

Universal Set: If there are some sets under consideration there happens to be a set, which is a super set of each one of the given sets, such a set is called the universal set denoted by U.

Complement of a Set: Let A be a sub-set of a universal set U. Then complement of A is denoted by A' or $U - A$ is the set of all those element of U which are not in A.

Binary Relation: Let A and B any two sets, then any sub-set the cartesian product $A \times B$ is called a binary relation from A to B.

Function: Any binary relation 'f' between two non-empty sets A and B such that:

(i) $Dom\ f = A$

(ii) First element in any two of the ordered pairs of f are not repeated, then f is called a function from A to B.

Unit-9 LINEAR GRAPHS

Plane: The walls of a class room, the black board, the top of the desk and the top of the table are all examples of a plane.

Cartesian Plane: A Cartesian plane consists of two number lines OX and OY intersecting at right angle at "O".

Ordinate: The perpendicular distance of a point from X-axis is called an ordinate.

Unit-10 BASIC STATISTICS

Histogram: When a bar chart is construed, so that the area of each bar is proportional to the number of items in each group is called a histogram.

Cumulative Frequency: Cumulative frequency is running total of class frequency.

Cumulative Frequency Polygon or (ogive): When the cumulative frequencies are plotted against the end point of their respective class intervals are joined together, the resultant graph is called a cumulative frequency polygon or ogive.

Measures of Central Tendency: These are summary statistics which measure the middle (or center) of the data.

Mean: To obtain the mean of a ungrouped data, all numbers in the set are added together and then the total is divided by the number of scores in that set.

Median: The middle value of data arranged in numerical order is called median.

Mode: The mode is the score which occurs most often in a set of data.

Arithmetic Mean: It is defined as the value obtained by dividing the sum of the values by their numbers. Thus the mean value of $x_1, x_2, \ldots\ldots\ldots, x_n$ denoted by $\bar{x}$ is :

$$\bar{x} = \frac{x_1, x_2, \ldots\ldots\ldots, x_n}{n}$$

Properties of arithmetic mean or mean: *(i)* The sum of deviations of values from their mean is zero. Symbolically: $\sum(x_i - \bar{x}) = 0$ or $\sum f_i(x_i - \bar{x}) = 0$

(ii) $x = \dfrac{\sum n\bar{x}}{\sum n}$

(iii) $\sum(x_i - a)^2$ is a minimum if and only if $a = \bar{x}$.

SYMBOLS

Symbol	Stands for	Symbol	Stands for
$<$	is less than	$\because$	because / as
$>$	is greater than	$\therefore$	therefore / so
$\leq$	is less than or equal to	$:$	ratio
$\geq$	is greater than or equal to	$::$	is proportional to
$=$	is equal to	∞	varies
$\neq$	is not equal to	$\vert$	tally mark
$\nless$	is not less than	Σ	summation
$\ngtr$	is not greater than	$\overline{AB}$	line segment AB
$\in$	belongs to / element	$\overrightarrow{AB}$	ray AB
$\forall$	for all	$\overleftrightarrow{AB}$	line
$\sqrt{\ }$	square root	$\angle$	angle
$\vert x\vert$	absolute value of x	$\triangle$	triangle
$\Rightarrow$	implies that	$\sim$	is similar to
$\Leftrightarrow$	if and only if	$\cong$	is congruent to
$\wedge$	and	$\approx$	is approximately equal to
$\cup$	union	$\Vert$	is parallel to
$\vee$	or	$\widehat{AB}$	arc AB
$\cap$	intersection	$\leftrightarrow$	correspondence
$\notin$	not element / not	U	universal sett

INDEX

A

B

C

D

E

F

G

R

S

T

U

V

W

Z

LOGARITHMS

	0	1	2	3	4	5	6	7	8	9	1	2	3	4	5	6	7	8	9
10	0000	0043	0086	0128	0170						4	9	13	17	21	26	30	34	38
						0212	0253	0294	0334	0374	4	8	12	16	20	24	28	32	36
11	0414	0453	0492	0531	0569						4	8	12	15	19	23	27	31	35
						0607	0645	0682	0719	0755	4	7	11	15	19	22	26	30	33
12	0792	0828	0864	0899	0934						3	7	11	14	18	21	25	28	32
						0969	1004	1038	1072	1106	3	7	10	14	17	20	24	27	31
13	1139	1173	1206	1239	1271						3	7	10	13	16	20	23	26	30
						1303	1335	1367	1399	1430	3	7	10	13	16	19	22	25	29
14	1461	1492	1523	1553	1584						3	6	9	12	15	19	22	25	28
						1614	1644	1673	1703	1732	3	6	9	12	15	17	20	23	26
15	1761	1790	1818	1847	1875						3	6	9	11	14	16	20	23	26
						1903	1931	1959	1987	2014	3	6	8	11	14	17	19	22	25
16	2041	2068	2095	2122	2148						3	5	8	11	14	17	19	22	24
						2175	2201	2227	2253	2279	3	5	8	10	13	16	18	21	23
17	2304	2330	2355	2380	2405						3	5	8	10	13	15	18	20	23
						2430	2455	2480	2504	2529	2	5	7	10	12	15	17	20	22
18	2553	2577	2601	2625	2648						2	5	7	9	12	14	16	19	21
						2672	2695	2718	2742	2765	2	5	7	9	11	14	16	18	21
19	2788	2810	2833	2856	2878						2	4	7	9	11	13	16	18	20
						2900	2923	2945	2967	2989	2	4	6	8	11	13	15	17	19
20	3010	3032	3054	3075	3096	3118	3139	3160	3181	3201	2	4	6	8	11	13	15	17	19
21	3222	3243	3263	3284	3304	3324	3345	3365	3385	3404	2	4	6	8	10	12	14	16	18
22	3424	3444	3464	3483	3502	3522	3541	3560	3579	3598	2	4	6	8	10	12	14	15	17
23	3617	3636	3655	3674	3692	3711	3729	3747	3766	3785	2	4	6	7	9	11	13	15	17
24	3802	3820	3838	3856	3874	3892	3909	3927	3945	3962	2	4	5	7	9	11	12	14	16
25	3979	3997	4014	4031	4048	4065	4082	4099	4116	4133	2	3	5	7	9	10	12	14	15
26	4150	4166	4183	4200	4216	4232	4249	4265	4281	4298	2	3	5	7	8	10	11	13	15
27	4314	4330	4346	4362	4378	4393	4409	4425	4440	4456	2	3	5	6	8	9	11	13	14
28	4472	4487	4502	4518	4533	4548	4564	4579	4594	4609	2	3	5	6	8	9	11	12	14
29	4624	4639	4654	4669	4683	4698	4713	4728	4742	4757	1	3	4	6	7	9	10	12	13
30	4771	4786	4800	4814	4829	4843	4857	4871	4886	4900	1	3	4	6	7	9	10	11	13
31	4914	4928	4942	4955	4969	4983	4997	5011	5024	5038	1	3	4	6	7	8	10	11	12
32	5051	5065	5079	5092	5105	5119	5132	5145	5159	5172	1	3	4	5	7	8	9	11	12
33	5185	5198	5211	5224	5237	5250	5263	5276	5289	5302	1	3	4	5	6	8	9	10	12
34	5315	5328	5340	5353	5366	5378	5391	5403	5416	5428	1	3	4	5	6	8	9	10	11
35	5441	5453	5465	5478	5490	5502	5514	5527	5539	5551	1	2	4	5	6	7	9	10	11
36	5563	5575	5587	5599	5611	5623	5635	5647	5658	5670	1	2	4	5	6	7	8	10	11
37	5682	5694	5705	5717	5729	5740	5752	5763	5775	5786	1	2	3	5	6	7	8	9	10
38	5798	5809	5821	5832	5843	5855	5866	5877	5888	5899	1	2	3	5	6	7	8	9	10
39	5911	5922	5933	5944	5955	5966	5977	5988	5999	6010	1	2	3	4	5	7	8	9	10
40	6021	6031	6042	6053	6064	6075	6085	6096	6107	6117	1	2	3	4	5	6	8	9	10
41	6128	6138	6149	6160	6170	6180	6191	6201	6212	6222	1	2	3	4	5	6	7	8	9
42	6232	6243	6253	6263	6274	6284	6294	6304	6314	6325	1	2	3	4	5	6	7	8	9
43	6335	6345	6355	6365	6375	6385	6395	6405	6415	6425	1	2	3	4	5	6	7	8	9
44	6435	6444	6454	6464	6474	6484	6493	6503	6513	6522	1	2	3	4	5	6	7	8	9
45	6532	6542	6551	6561	6571	6580	6590	6599	6609	6618	1	2	3	4	5	6	7	8	9
46	6628	6637	6646	6656	6665	6675	6684	6693	6702	6712	1	2	3	4	5	6	7	7	8
47	6721	6730	6739	6749	6758	6767	6776	6785	6794	6803	1	2	3	4	5	5	6	7	8
48	6812	6821	6830	6839	6848	6857	6866	6875	6884	6893	1	2	3	4	4	5	6	7	8
49	6902	6911	6920	6928	6937	6946	6955	6964	6972	6981	1	2	3	4	4	5	6	7	8

LOGARITHMS

	0	1	2	3	4	5	6	7	8	9	1	2	3	4	5	6	7	8	9
50	6990	6998	7007	7016	7024	7033	7042	7050	7059	7067	1	2	3	3	4	5	6	7	8
51	7076	7084	7093	7101	7110	7118	7126	7135	7143	7152	1	2	3	3	4	5	6	7	8
52	7160	7168	7177	7185	7193	7202	7210	7218	7226	7235	1	2	2	3	4	5	6	7	7
53	7243	7251	7259	7267	7275	7284	7292	7300	7308	7316	1	2	2	3	4	5	6	6	7
54	7324	7332	7340	7348	7356	7364	7372	7380	7388	7396	1	2	2	3	4	5	6	6	7
55	7404	7412	7419	7427	7435	7443	7451	7459	7466	7474	1	2	2	3	4	5	5	6	7
56	7482	7490	7497	7505	7513	7520	7528	7536	7543	7551	1	2	2	3	4	5	5	6	7
57	7559	7566	7574	7582	7589	7597	7604	7612	7619	7627	1	2	2	3	4	5	5	6	7
58	7634	7642	7649	7657	7664	7672	7679	7686	7694	7701	1	1	2	3	4	4	5	6	7
59	7709	7716	7723	7731	7738	7745	7752	7760	7767	7774	1	1	2	3	4	4	5	6	7
60	7782	7789	7796	7803	7810	7818	7825	7832	7839	7846	1	1	2	3	4	4	5	6	6
61	7853	7860	7868	7875	7882	7889	7896	7903	7910	7917	1	1	2	3	4	4	5	6	6
62	7924	7931	7938	7945	7952	7959	7966	7973	7980	7987	1	1	2	3	3	4	5	6	6
63	7993	8000	8007	8014	8021	8028	8035	8041	8048	8055	1	1	2	3	3	4	5	5	6
64	8062	8069	8075	8082	8089	8096	8102	8109	8116	8122	1	1	2	3	3	4	5	5	6
65	8129	8136	8142	8149	8156	8162	8169	8176	8182	8189	1	1	2	3	3	4	5	5	6
66	8195	8202	8209	8215	8222	8228	8235	8241	8248	8254	1	1	2	3	3	4	5	5	6
67	8261	8267	8274	8280	8287	8293	8299	8306	8312	8319	1	1	2	3	3	4	5	5	6
68	8325	8331	8338	8344	8351	8357	8363	8370	8376	8382	1	1	2	3	3	4	4	5	6
69	8388	8395	8401	8407	8414	8420	8426	8432	8439	8445	1	1	2	2	3	4	4	5	6
70	8451	8457	8463	8470	8476	8482	8488	8494	8500	8506	1	1	2	2	3	4	4	5	6
71	8513	8519	8525	8531	8537	8543	8549	8555	8561	8567	1	1	2	2	3	4	4	5	5
72	8573	8579	8585	8591	8597	8603	8609	8615	8621	8627	1	1	2	2	3	4	4	5	5
73	8633	8639	8645	8651	8657	8663	8669	8675	8681	8686	1	1	2	2	3	4	4	5	5
74	8692	8698	8704	8710	8716	8722	8727	8733	8738	8745	1	1	2	2	3	4	4	5	5
75	8751	8756	8762	8768	8774	8779	8785	8791	8797	8802	1	1	2	2	3	3	4	5	5
76	8808	8814	8820	8825	8831	8837	8842	8848	8854	8859	1	1	2	2	3	3	4	5	5
77	8865	8871	8876	8882	8887	8893	8899	8904	8910	8915	1	1	2	2	3	3	4	4	5
78	8921	8927	8932	8938	8943	8949	8954	8960	8965	8971	1	1	2	2	3	3	4	4	5
79	8976	8982	8987	8993	8998	9004	9009	9015	9020	9025	1	1	2	2	3	3	4	4	5
80	9031	9036	9042	9047	9053	9058	9063	9069	9074	9079	1	1	2	2	3	3	4	4	5
81	9085	9090	9096	9101	9106	9112	9117	9122	9128	9133	1	1	2	2	3	3	4	4	5
82	9138	9143	9149	9154	9159	9165	9170	9175	9180	9186	1	1	2	2	3	3	4	4	5
83	9191	9196	9201	9206	9212	9217	9222	9227	9232	9238	1	1	2	2	3	3	4	4	5
84	9243	9248	9253	9258	9263	9269	9274	9279	9284	9289	1	1	2	2	3	3	4	4	5
85	9294	9299	9304	9309	9315	9320	9325	9330	9335	9340	1	1	2	2	3	3	4	4	5
86	9345	9350	9355	9360	9365	9370	9375	9380	9385	9390	1	1	2	2	3	3	4	4	5
87	9395	9400	9405	9410	9415	9420	9425	9430	9435	9440	0	1	1	2	2	3	3	4	4
88	9445	9450	9455	9460	9465	9469	9474	9479	9484	9489	0	1	1	2	2	3	3	4	4
89	9494	9499	9504	9509	9513	9518	9523	9528	9533	9538	0	1	1	2	2	3	3	4	4
90	9542	9547	9552	9557	9562	9566	9571	9576	9581	9586	0	1	1	2	2	3	3	4	4
91	9590	9595	9600	9605	9609	9614	9619	9624	9628	9633	0	1	1	2	2	3	3	4	4
92	9638	9643	9647	9652	9657	9661	9666	9671	9675	9680	0	1	1	2	2	3	3	4	4
93	9685	9689	9694	9699	9603	9708	9713	9717	9722	9727	0	1	1	2	2	3	3	4	4
94	9731	9736	9741	9745	9750	9754	9759	9763	9768	9773	0	1	1	2	2	3	3	4	4
95	9777	9782	9786	9791	9795	9800	9805	9809	9814	9818	0	1	1	2	2	3	3	4	4
96	9823	9827	9832	9836	9841	9845	9850	9854	9859	9863	0	1	1	2	2	3	3	4	4
97	9868	9872	9877	9881	9886	9890	9894	9899	9903	9908	0	1	1	2	2	3	3	4	4
98	9912	9917	9921	9926	9930	9934	9939	9943	9948	9952	0	1	1	2	2	3	3	4	4
99	9956	9961	9965	9969	9974	9978	9983	9987	9991	9996	0	1	1	2	2	3	3	3	4

ANTI LOGARITHMS

	0	1	2	3	4	5	6	7	8	9	1	2	3	4	5	6	7	8	9
.00	1000	1002	1005	1007	1009	1012	1014	1016	1019	1021	0	0	1	1	1	1	2	2	2
.01	1023	1026	1027	1030	1033	1035	1038	1040	1042	1045	0	0	1	1	1	1	2	2	2
.02	1047	1050	1052	1054	1057	1059	1062	1064	1067	1069	0	0	1	1	1	1	2	2	2
.03	1072	1074	1076	1079	1081	1084	1086	1089	1091	1094	0	0	1	1	1	1	2	2	2
.04	1096	1099	1102	1104	1107	1109	1112	1114	1117	1119	0	0	1	1	1	2	2	2	2
.05	1122	1125	1127	1130	1132	1135	1138	1140	1143	1146	0	1	1	1	1	2	2	2	2
.06	1148	1151	1153	1156	1159	1161	1164	1167	1169	1172	0	1	1	1	1	2	2	2	2
.07	1175	1178	1180	1183	1186	1189	1191	1194	1197	1199	0	1	1	1	1	2	2	2	2
.08	1202	1205	1208	1211	1213	1216	1219	1222	1225	1227	0	1	1	1	1	2	2	2	3
.09	1230	1235	1236	1239	1242	1245	1247	1250	1253	1256	0	1	1	1	1	2	2	2	3
.10	1259	1262	1265	1268	1271	1274	1276	1279	1282	1285	0	1	1	1	1	2	2	2	3
.11	1288	1291	1294	1297	1300	1303	1306	1309	1312	1315	0	1	1	1	2	2	2	2	3
.12	1318	1321	1324	1327	1330	1334	1337	1340	1343	1346	0	1	1	1	2	2	2	2	3
.13	1349	1352	1355	1358	1361	1365	1368	1371	1374	1377	0	1	1	1	2	2	2	3	3
.14	1380	1384	1387	1390	1393	1396	1400	1403	1406	1409	0	1	1	1	2	2	2	3	3
.15	1413	1416	1419	1422	1426	1429	1432	1435	1439	1442	0	1	1	1	2	2	2	3	3
.16	1445	1449	1452	1455	1459	1462	1466	1469	1472	1476	0	1	1	1	2	2	2	3	3
.17	1479	1483	1486	1489	1493	1496	1500	1503	1507	1510	0	1	1	1	2	2	2	3	3
.18	1514	1517	1521	1524	1528	1531	1535	1538	1542	1545	0	1	1	1	2	2	2	3	3
.19	1549	1552	1556	1560	1563	1567	1570	1574	1578	1581	0	1	1	1	2	2	3	3	3
.20	1585	1589	1592	1596	1600	1603	1607	1611	1614	1618	0	1	1	1	2	2	3	3	3
.21	1622	1626	1629	1633	1637	1641	1644	1648	1652	1656	0	1	1	2	2	2	3	3	3
.22	1660	1663	1667	1671	1675	1679	1683	1687	1690	1694	0	1	1	2	2	2	3	3	3
.23	1698	1702	1706	1710	1714	1718	1722	1726	1730	1734	0	1	1	2	2	2	3	3	4
.24	1738	1742	1746	1750	1754	1758	1762	1766	1770	1774	0	1	1	2	2	2	3	3	4
.25	1778	1782	1786	1791	1795	1799	1803	1807	1811	1816	0	1	1	2	2	2	3	3	4
.26	1820	1824	1828	1832	1837	1841	1845	1849	1854	1858	0	1	1	2	2	3	3	3	4
.27	1862	1866	1871	1875	1879	1884	1888	1892	1897	1901	0	1	1	2	2	3	3	3	4
.28	1905	1910	1914	1919	1923	1928	1932	1936	1941	1945	0	1	1	2	2	3	3	4	4
.29	1950	1954	1959	1963	1968	1972	1977	1982	1986	1991	0	1	1	2	2	3	3	4	4
.30	1995	2000	2004	2009	2014	2018	2023	2028	2032	2037	0	1	1	2	2	3	3	4	4
.31	2042	2046	2051	2056	2061	2065	2070	2075	2080	2084	0	1	1	2	2	3	3	4	4
.32	2089	2094	2099	2104	2109	2113	2118	2123	2128	2133	0	1	1	2	2	3	3	4	4
.33	2138	2143	2148	2153	2158	2163	2168	2173	2178	2183	0	1	1	2	2	3	3	4	4
.34	2188	2193	2198	2203	2208	2213	2218	2223	2228	2234	1	1	2	2	3	3	4	4	5
.35	2239	2244	2249	2254	2259	2265	2270	2275	2280	2286	1	1	2	2	3	3	4	4	5
.36	2291	2296	2301	2307	2312	2317	2323	2328	2333	2339	1	1	2	2	3	3	4	4	5
.37	2344	2350	2355	2360	2366	2371	2377	2382	2388	2393	1	1	2	2	3	3	4	4	5
.38	2399	2404	2410	2415	2421	2427	2432	2438	2443	2449	1	1	2	2	3	3	4	4	5
.39	2455	2460	2466	2472	2477	2483	2489	2495	2500	2506	1	1	2	2	3	3	4	5	5
.40	2512	2518	2523	2529	2535	2541	2547	2553	2559	2564	1	1	2	2	3	4	4	5	5
.41	2570	2576	2582	2588	2594	2600	2606	2612	2618	2624	1	1	2	2	3	4	4	5	5
.42	2630	2636	2642	2649	2655	2661	2667	2673	2679	2685	1	1	2	2	3	4	4	5	6
.43	2692	2698	2704	2710	2716	2723	2729	2735	2742	2748	1	1	2	3	3	4	4	5	6
.44	2754	2761	2767	2773	2780	2786	2793	2799	2805	2812	1	1	2	3	3	4	4	5	6
.45	2818	2825	2831	2838	2844	2851	2858	2864	2871	2877	1	1	2	3	3	4	5	5	6
.46	2884	2891	2897	2904	2911	2917	2924	2931	2938	2944	1	1	2	3	3	4	5	5	6
.47	2951	2958	2965	2972	2979	2985	2992	2999	3006	3013	1	1	2	3	3	4	5	5	6
.48	3020	3027	3034	3041	3048	3055	3062	3069	3076	3083	1	1	2	3	4	4	5	6	6
.49	3090	3097	3105	3112	3119	3126	3133	3141	3148	3155	1	1	2	3	4	4	5	6	6

ANTI LOGARITHMS

	0	1	2	3	4	5	6	7	8	9	1	2	3	4	5	6	7	8	9
.50	3162	3170	3177	3184	3192	3199	3206	3214	3221	3228	1	1	2	3	4	4	5	6	7
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.53	3388	3396	3404	3412	3420	3428	3436	3443	3451	3459	1	2	2	3	4	5	6	6	7
.54	3467	3475	3483	3491	3499	3508	3516	3524	3532	3540	1	2	2	3	4	5	6	6	7
.55	3548	3556	3565	3573	3581	3589	3597	3606	3614	3622	1	2	2	3	4	5	6	7	7
.56	3631	3639	3648	3656	3664	3673	3681	3690	3698	3707	1	2	3	3	4	5	6	7	8
.57	3715	3724	3733	3741	3750	3758	3767	3776	3784	3793	1	2	3	3	4	5	6	7	8
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.61	4074	4083	4093	4102	4111	4121	4130	4140	4150	4159	1	2	3	4	5	6	7	8	9
.62	4169	4178	4188	4198	4207	4217	4227	4236	4246	4256	1	2	3	4	5	6	7	8	9
.63	4266	4276	4285	4295	4305	4315	4325	4335	4345	4355	1	2	3	4	5	6	7	8	9
.64	4365	4375	4385	4395	4406	4416	4426	4436	4446	4457	1	2	3	4	5	6	7	8	9
.65	4467	4477	4487	4498	4508	4519	4529	4539	4550	4560	1	2	3	4	5	6	7	8	9
.66	4571	4581	4592	4603	4613	4624	4634	4645	4656	4667	1	2	3	4	5	6	7	9	10
.67	4677	4688	4699	4710	4721	4732	4742	4753	4764	4775	1	2	3	4	5	7	8	9	10
.68	4786	4797	4808	4819	4831	4842	4853	4864	4875	4887	1	2	3	4	6	7	8	9	10
.69	4898	4909	4920	4932	4943	4955	4966	4977	4989	5000	1	2	3	5	6	7	8	9	10
.70	5012	5023	5035	5047	5058	5070	5082	5093	5105	5117	1	2	4	5	6	7	8	9	11
.71	5129	5140	5152	5164	5176	5188	5200	5212	5224	5236	1	2	4	5	6	7	8	10	11
.72	5248	5260	5272	5284	5297	5309	5321	5333	5346	5358	1	2	4	5	6	7	9	10	11
.73	5370	5383	5395	5408	5420	5433	5445	5458	5470	5483	1	3	4	5	6	8	9	10	11
.74	5495	5508	5521	5534	5546	5559	5572	5585	5598	5610	1	3	4	5	6	8	9	10	12
.75	5623	5636	5649	5662	5675	5689	5702	5715	5728	5741	1	3	4	5	7	8	9	10	12
.76	5754	5768	5781	5794	5808	5821	5834	5848	5861	5875	1	3	4	5	7	8	9	11	12
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.78	6026	6039	6053	6067	6081	6095	6109	6124	6138	6152	1	3	4	6	7	8	10	11	13
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.81	6457	6471	6486	6501	6516	6531	6546	6561	6577	6592	2	3	5	6	8	9	11	12	14
.82	6607	6622	6637	6653	6668	6683	6699	6714	6730	6745	2	3	5	6	8	9	11	12	14
.83	6761	6776	6792	6808	6823	6839	6855	6871	6887	6902	2	3	5	6	8	9	11	13	14
.84	6918	6934	6950	6966	6982	6998	7015	7031	7047	7063	2	3	5	6	8	10	11	13	15
.85	7079	7096	7112	7129	7145	7161	7178	7194	7211	7228	2	3	5	7	8	10	12	13	15
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.87	7413	7430	7447	7464	7482	7499	7516	7534	7551	7568	2	3	5	7	9	10	12	14	16
.88	7586	7603	7621	7638	7656	7674	7691	7709	7727	7745	2	4	5	7	9	11	12	14	16
.89	7762	7780	7798	7816	7834	7852	7870	7889	7907	7925	2	4	5	7	9	11	13	14	16
.90	7943	7962	7980	7998	8017	8035	8054	8072	8091	8110	2	4	6	7	9	11	13	15	17
.91	8128	8147	8166	8185	8204	8222	8241	8260	8279	8299	2	4	6	8	9	11	13	15	17
.92	8318	8337	8356	8375	8395	8414	8433	8453	8472	8492	2	4	6	8	10	12	14	15	17
.93	8511	8531	8551	8570	8590	8610	8630	8650	8670	8690	2	4	6	8	10	12	14	16	18
.94	8710	8730	8750	8770	8790	8810	8831	8851	8872	8892	2	4	6	8	10	12	14	16	18
.95	8913	8933	8954	8974	8995	9016	9036	9057	9078	9099	2	4	6	8	10	12	15	17	19
.96	9120	9141	9162	9183	9204	9226	9247	9268	9290	9311	2	4	6	8	11	13	15	17	19
.97	9333	9354	9376	9397	9419	9441	9462	9484	9506	9528	2	4	7	9	11	13	15	17	20
.98	9550	9572	9594	9616	9638	9661	9683	9705	9727	9750	2	4	7	9	11	13	16	18	20
.99	9772	9795	9817	9840	9863	9886	9908	9931	9954	9977	2	5	7	9	11	14	16	18	20