Unit 7

OSCILLATION

Major Concepts **(23 PERIODS)**

- Simple Harmonic Motion (SHM)
- Circular motion and SHM
- Practical SHM system (mass spring and simple pendulum)
- Energy conservation in SHM
- Free and forced oscillations
- Resonance
- Damped oscillations

Conceptual Linkage

This chapter is built on Circular Motion Physics XI Oscillation & Waves Physics XI

Students Learning Outcomes

After studying this unit, the students will be able to:

- Describe simple examples of free oscillations.
- Describe necessary conditions for execution of simple harmonic motions.
- Describe that when an object moves in a circle, the motion of its projection on the diameter of the circle is SHM.
- Define the terms amplitude, period, frequency, angular frequency and phase difference and express the period in terms of both frequency and angular frequency.
- Identify and use the equation; $a = -\omega^2 x$ as the defining equation of SHM.
- Prove that the motion of mass attached to a spring is SHM.
- Describe the interchanging between kinetic energy and potential energy during SHM.
- Analyze the motion of a simple pendulum is SHM and calculate its time period.
- Describe practical examples of free and forced oscillations (resonance).
- Describe graphically how the amplitude of a forced oscillation changes with frequency near to the natural frequency of the system.
- Describe practical examples of damped oscillations with particular reference to the efforts of the degree of damping and the importance of critical damping in cases such as a car suspension system.
- Describe qualitatively the factors which determine the frequency response and sharpness of the resonance.

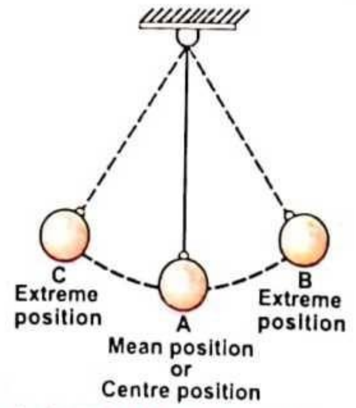
INTRODUCTION

Besides translational and rotational motion, there is another important kind of motion that is vibrational motion which has too many applications in physics as well as in our daily life. This kind of motion of a body is a to and fro motion about its mean position and its nature is a periodic motion because the oscillating body repeats itself after a regular intervals of time. Some examples of oscillations are given below.

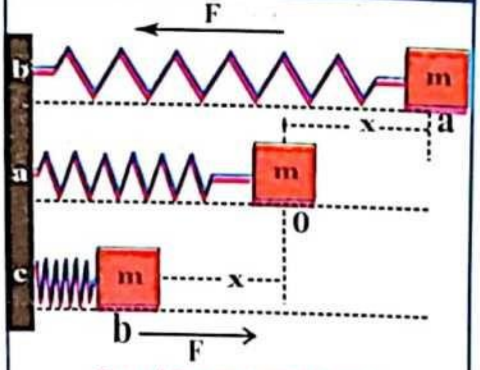
(i) Swinging of a simple pendulum when it is displaced from its mean position and is made to free.

(ii) Motion of a body attached to a spring when it is pulled and then released.

(iii) Vibration of prongs of the tuning fork when it is struck on a rubber paid.



Swinging of a simple pendulum



To and fro motion of a body attached to a spring

All the bodies that undergo vibrational or oscillational motion have an equilibrium position or mean position. When the body is displaced from this mean position then there is a restoring force which brings it back to its equilibrium position and it causes of vibration or oscillation motion of the body.

The detailed study of vibrational motion helps us in the understanding of waves, sounds, light and alternating current because it has been observed that vibrating bodies produce waves. For example, a violin string produces sound waves in air.


Vibrating tuning fork

Resonance is a striking phenomenon which is related with vibrational motion and it plays a dynamic role in communication system because maximum communication energy transfer is processed by transmitter and receiver due to the resonance phenomenon.

Though many systems cannot operate without resonance but it should be avoided in some cases such as aeroplane wings or helicopter rotor and suspension bridges etc.

In this chapter we will study not only various parameters related to an oscillating body but will also prove that the motions of a particle along a circle, a body attached to a string and a simple pendulum are simple harmonic motion.

7.1 SIMPLE HARMONIC MOTION (SHM)

The back and forth motion of a body that it repeats in equal interval of time along the same line is called periodic motion. On the other hand, simple harmonic motion is the most important type of the periodic motion and it occurs when the restoring force is directly proportional to the displacement from an equilibrium position. It can be explained with the example of a body of mass 'm' attached to a spring which oscillates about equilibrium position 'O' on a horizontal frictionless surface as shown in Fig.7.1. Consider a force 'F' that is applied to displace the body from its equilibrium position 'O' to an extreme position through a distance 'x'.

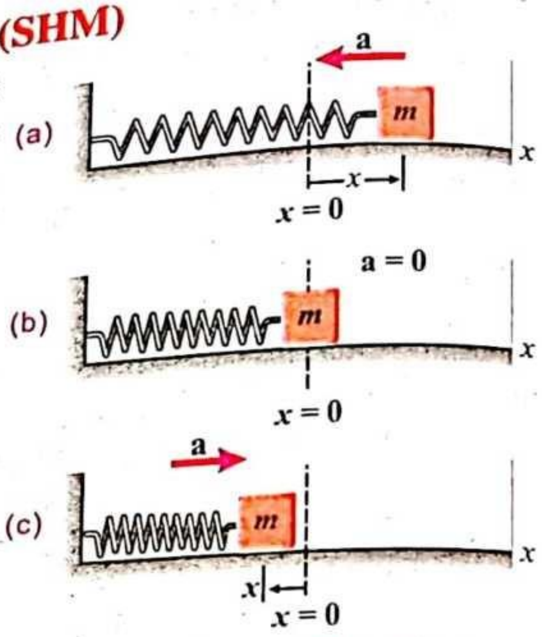


Fig.7.1: Mass attached with spring during its SHM
(a) At the right extreme position
(b) At mean position
(c) At the left extreme position

According to Hook's law, the applied force is equal to kx, Where 'k' is a constant and is called spring constant, and it has the dimensions of force per unit length (Nm^{-1}). Due to the elasticity of the spring, an elastic restoring force $(-kx)$ acts on the body whose magnitude is equal to applied force and its direction is towards the mean position and there is also an acceleration which is produced by such restoring force. This acceleration causes simple harmonic motion in the body and is directly proportional to the displacement and is always directed towards the mean position. These two conditions are known as the conditions that must be obeyed by a body in order to execute simple harmonic motion.

According to Hook's law the elastic restoring force is given by:

$$F = -kx \;\ldots\ldots(7.1)$$

According to Newton's 2^{nd} law of motion

$$F = ma \;\ldots\ldots(7.2)$$

Comparing equation (7.1) and equation (7.2)

$$ma = -kx$$

$$a = -\left(\frac{k}{m}\right)x \;\ldots\ldots(7.3)$$

As the ratio (k/m) is a constant therefore,

$$a = -(\text{Constant})x$$

$$a \propto -x$$

This is the mathematical form of simple harmonic motion. It states that the acceleration of the body executing simple harmonic motion (SHM) is directly proportional to the displacement and negative sign shows that it is directed toward its mean position.

Example 7.1

A body of mass 0.25 kg is connected to a spring and it is oscillating on a horizontal frictionless surface. If the maximum displacement of body is 20cm and the spring constant is 10 N m^{-1} then what is the acceleration of the body?

POINT TO PONDER

Can a linear motion of a body be SHM?

Solution: $m = 0.25$ kg

$k = 10 \text{ N m}^{-1}$

$x = 20 \text{ cm} = 0.2 \text{ m}$

$a = ?$

$$a = -\left(\frac{k}{m}\right)x$$

$$a = -\left(\frac{10}{0.25}\right)0.2 = -8 \text{ m s}^{-2}$$

Negative sign shows that the motion of body is directed towards its mean position.

7.1.1 Characteristic of simple harmonic motion

Simple harmonic motion is a special kind of periodic motion. It can be represented graphically by demonstrating an experiment of mass spring system. The experimental set up consists of a block of mass 'm' attached with a spring which is hanging vertically and remains at its equilibrium position 'O' as shown in Fig. 7.2.

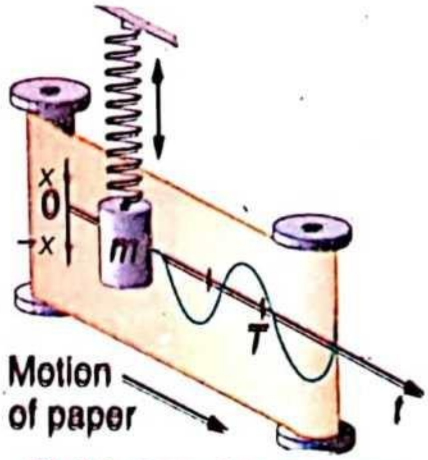


Fig.7.2: Pen and paper arrangement to draw a graph of an oscillating body

A sheet of paper with a suitable time scale is placed behind the block which is moving at a constant speed from right to left. There is also a pen which is attached with the vibrating mass which lightly touches the paper in order to record the variations in displacement with time during the oscillation of mass.

When the block is displaced downward from its mean position to its extreme position at a distance 'x' and is made to free then it starts oscillation. As a result, displacement against time appears on the paper in the form of sinusoidal-wave which is known as wave form of simple harmonic motion as shown in Fig. 7.3.

The various parameters related with simple harmonic motion are summarized as:

I) Instantaneous displacement

In vibrational motion, the distance from the mean position at any instant is known as instantaneous displacement. It is zero at the instant when the body is at mean position and it is maximum at the extreme position.

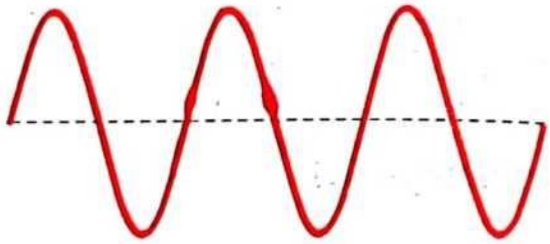


Fig.7.3: A Sine wave shape of an oscillating body

II) Amplitude

In Vibrational motion, the maximum distance from the mean position to either extreme position is known as amplitude. The SI unit of amplitude is metre.

III) Vibration

One complete round trip of a body during its vibrational motion is called vibration. For example, when the body starts its motion from its first extreme position ($-x$) to the second extreme position (x) and then from the second extreme position (x) to the first extreme position ($-x$) crossing the mean position (o) is called one vibration as shown in Fig. 7.2.

IV) Time Period

Time period is defined as the time taken to complete one vibration or one cycle. It is represented by 'T' and its SI unit is second 's'.

V) Frequency

Frequency is defined as the number of vibrations completed by the vibrating body in one second. It is expressed in terms of the reciprocal of time period that is;

$$f = \frac{1}{T} \quad ......(7.4)$$

The unit of frequency is hertz (Hz) and it is equal to per second. The dimensional formula of frequency is $[M^oL^oT^{-1}]$.

VI) Angular Frequency

Angular frequency is defined as the number of revolutions per unit time. It is represented by 'ω' and it can be expressed as;

$$\omega = \frac{\theta}{t}$$

Now for one revolution $\theta = 2\pi$ radians and $t = T$ (time period)

$$\omega = \frac{2\pi}{T}$$

As $$T = \frac{1}{f}$$

Therefore, $$\omega = 2\pi f \quad ......(7.5)$$

The SI unit of angular frequency is rad.s^{-1} and its dimensional formula is $[M^oL^oT^{-1}]$

Example 7.2

A mass connected to a spring makes 15 vibrations in 45 second. Calculate its period and frequency.

POINT TO PONDER

Every vibrating body produces a sound. Does a simple pendulum also produce a sound?

Solution:

Numbers of vibration = 15

Time for 15 vibrations = 45 s

$$T = ?$$

$$f = ?$$

$$\text{Time period (T)} = \frac{\text{given time}}{\text{No. of vibs.}}$$

$$\text{Time period (T)} = \frac{45}{15} = 3 \text{ s}$$

$$\text{Frequency} = f = \frac{1}{T}$$

$$\text{Frequency} = f = \frac{1}{3} = 0.333 \text{ Hz}$$

7.2 CIRCULAR MOTION AND SIMPLE HARMONIC MOTION

To study the simple harmonic motion, consider a turntable of radius 'r' with a ball attached to its rim. A beam of light casts a shadow of the ball on the screen as shown in Fig.7.4.

When the turntable rotates with constant angular speed 'ω' then the ball also moves along it with uniform circular motion. Its shadow on the screen oscillates executing to and fro motion across the screen in the form of simple harmonic motion like a body attached to a spring.

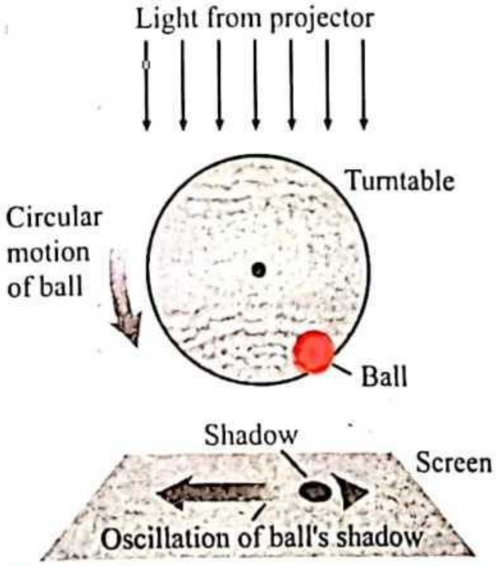


Fig.7.4: The oscillation of the shadow of the ball on screen. The ball is attached with uniformly rotating turn table.

Now we can study the motion of the ball along the circumference of the turn table and its resulting shadow on the screen along the diameter for one complete cycle.

Let the projection of the ball be on the mean position 'O' at t = 0 then after some instant $t = \frac{T}{4}$, the projection will be on the left extreme position 'A'.

Similarly, after instant $t = \frac{T}{2}$ the projection is again at the mean position 'O', at $t = \frac{3T}{4}$, the projection is on the right extreme position 'B'. Finally at $t = T$, the projection reaches at its starting point i.e. the mean position O. Hence, one cycle is completed. In the same way, the next cycles will also give the same result. When the graph between displacement and time is plotted then we have a sinusoidal wave as shown in Fig.7.5. This example clearly indicates that when an object moves along the circumference of a circle, its projection on the diameter of the circle executes S.H.M. The parameters such as displacement, velocity, acceleration, time period and phase of the S.H.M by the projection of the particle are explained below.

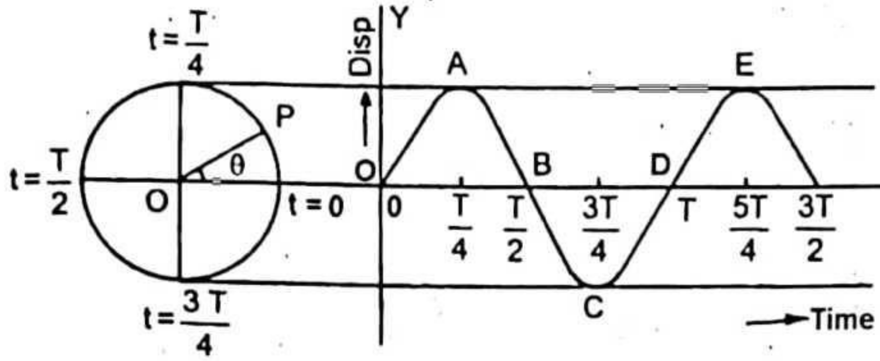


Fig.7.5: The wave shape of the projection of the ball executing S.H.M

7.2.1 Quantitative Analysis

Consider a motion of particle 'P' along the circumference of circle of radius 'r' with uniform angular velocity 'ω'. Its linear velocity at point 'P' is along the tangent ($v_p = r\omega$) and its acceleration a_P is directed towards the centre of circle as shown in Fig. 7.6. The value of acceleration is given as;

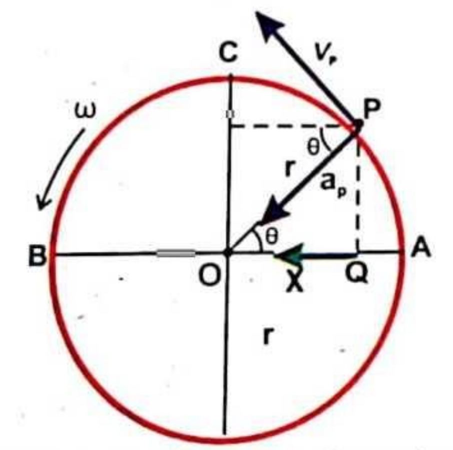


Fig.7.6: A particle which is moving along a circular path of circle with uniform angular velocity and its linear velocity v_p is tangent.

$$a_p = \frac{v_P^2}{r} \quad \because v = r\omega$$

$$a_p = \frac{r^2\omega^2}{r}$$

$$a_p = r\omega^2 \quad ......(7.6)$$

The particle 'P' is making an angle 'θ' and its projection point on a diameter is '*Q*' which are shown in Fig. 7.7.

When the particle moves along the circumference, its projection '*Q*' also starts its motion along the diameter from point *A* to point *B* then point '*B*' to point '*A*' about the mean position '*O*' performing simple harmonic motion.

Displacement

At time t = 0 the particle 'P' subtends an angle $\angle P\hat{O}Q = \theta = \omega t$ with OQ and the displacement of Q is 'x' which is equal to OQ as shown in Fig. 7.7.

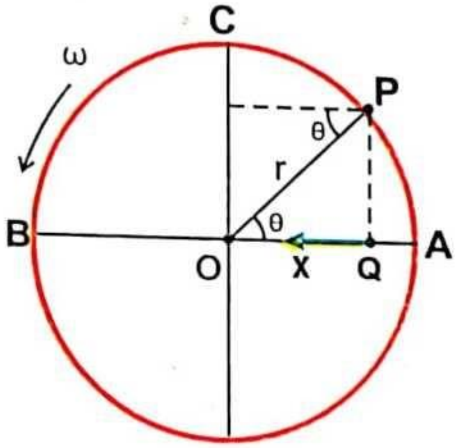


Fig.7.7: Displacement of the projection Q of particle P along the diameter AB.

Considering triangle POQ

$$\frac{OQ}{OP} = \cos\theta$$

$$\frac{x}{r} = \cos\omega t$$

$$x = r\cos\omega t \quad ......(7.7)$$

Eq. (7.7) gives the instantaneous displacement of point Q which is executing simple harmonic motion (SHM).

Velocity

In Fig. 7.8, the line $\overline{PR}$ is the horizontal component of velocity v_P of the particle and it is parallel to the diameter 'AB' of the circle. Therefore,

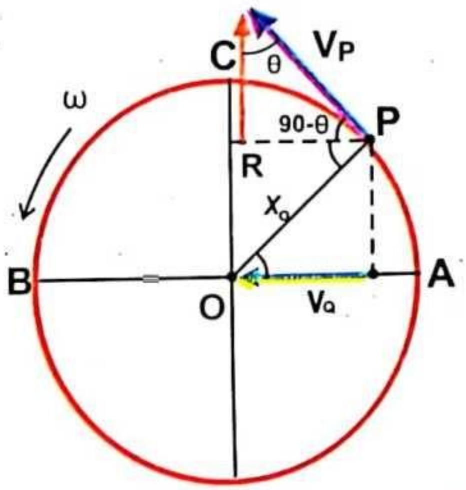


Fig.7.8: Velocity of the projection Q of particle P along the diameter AB.

$$v_Q = (v_P)_x$$

$$v_Q = v_P \cos(90° - \theta)$$

Since $v_P = r\omega$ and $\cos(90° - \theta) = \sin\theta$

$$v_Q = r\omega\sin\theta \,......(7.8)$$

But $\sin^2\theta + \cos^2\theta = 1$

$$\sin\theta = \sqrt{1-\cos^2\theta}$$

Equation 7.8 becomes

$$v_Q = r\omega\sqrt{1-\cos^2\theta}$$

$$v_Q = r\omega\sqrt{1-\frac{x^2}{r^2}} \quad \left[\because \cos\theta = \frac{x}{r}\right]$$

$$v_Q = \omega\sqrt{r^2 - x^2} \quad ......(7.9)$$

It may be noted that at mean position $x = 0$ and velocity is maximum.

$$v_m = v_o = r\omega \quad ......(7.10)$$

Acceleration

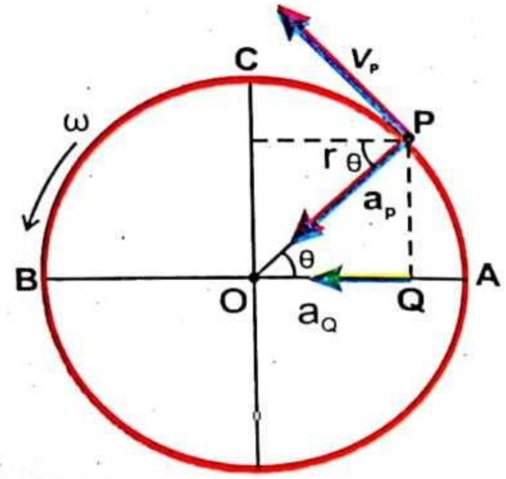


Fig.7.9: A particle which is moving along a circular path of circle its acceleration is directed toward the centre.

Acceleration 'a_P' of the particle at point 'P' is directed towards the centre of the circle as shown in Fig.7.9. The horizontal component of a_P is along the diameter. Thus the acceleration of projection 'Q' is equal to the horizontal component of a_P.

$$a_Q = -(a_P)_x$$

$$a_Q = -a_P \cos\theta$$

The negative sign indicates that the direction of acceleration is always directed towards the mean position.

$$a_Q = -r\omega^2\left(\frac{x}{r}\right) \qquad \left[\because a_P = r\omega^2, \cos\theta = \frac{x}{r}\right]$$

$$a_Q = -\omega^2 x \quad ......(7.11)$$

As particle is moving in the circle with uniform angular frequency (ω) therefore, Eq. (7.11) can be rewritten as;

$$a_Q \propto -x$$

This expression is the mathematical condition of S.H.M i.e. acceleration is directly proportional to the displacement and negative sign shows that its direction is towards the mean position. Therefore, it is concluded that when a particle is moving along a circumference of a circle then its projection executes S.H.M.

Time Period

It is defined as the time is required to complete one vibration by 'Q' from point A to B and then B to A. This is the same time in which the particle completes one revolution. It is denoted by 'T'.

Using the relationship $\omega = \dfrac{\theta}{t}$

For one complete cycle, $\theta = 2\pi$ radians and $t = T$ (time period).

$$\omega = \frac{2\pi}{T}$$

$$T = \frac{2\pi}{\omega}$$

Phase

The phase of an oscillating body determines its positions and direction of motion at a particular instant.

Consider the particle that moves along the circular path of a circle. Let at time $t = 0$ the particle is at point 'P' and its position vector OP makes an angle 'φ' with OA. After some time 't' the particle is at point P′ as OP makes angle $\theta = \omega t$ with OP. This angle determines both position and direction of the body at any instant and it is called phase angle which varies with time.

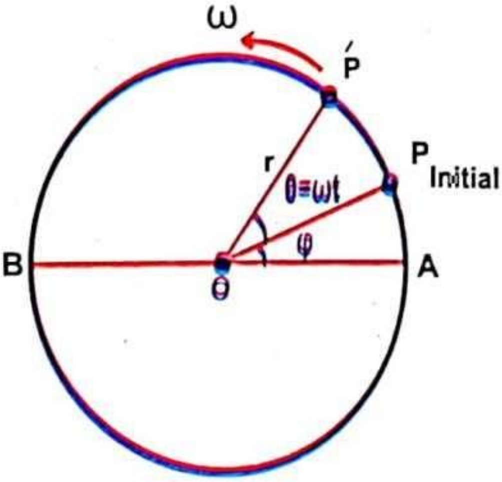


Fig.7.10: Phase angle and phase constant of a particle which is moving along a circle.

Now the total angle at point P is $\theta + \varphi$ is shown in Fig. 7.10. At time $t = 0$, phase $=+\varphi$. Sometimes at $t = 0$ the phase $= -\varphi$. In general the phase can be expressed as $\theta \pm \varphi$ or $\omega t \pm \varphi$, where φ is a phase constant which represents the initial position of a particles and it remains constant.

Example 7.3

A particle vibrates according to the equation $x = 0.3\cos 16t$. Find amplitude, frequency and its position at $t = 0$.

Solution:

As given $\quad x = 0.3\cos 16t \quad ......(7.12)$

The general equation for displacement of vibrating body is.

$$x = x_o \cos\omega t \quad ......(7.13)$$

Comparing equation (7.12) and equation (7.13)

$x_o = 0.3\text{m}$ and $\omega = 16$ vib/s

But $\omega = 2\pi f$

$$f = \frac{\omega}{2\pi}$$

$$f = \frac{16}{2(3.14)}$$

$$f = 2.55 \text{ Hz}$$

> **FOR YOUR INFORMATION**
>
> In SHM, the acceleration a is proportional to the displacement x but opposite in direction, and the two quantities are related by the square of the angular frequency ω.

Position at $t = 0$

$$x = 0.3\cos 0^\circ = 0.3(1) \qquad \therefore \cos 0^\circ = 1$$

$$x = 0.3 \text{ m}$$

7.3 MASS-SPRING SYSTEM AND S.H.M

Consider a block of mass 'm' which is attached to one end of a horizontal spring. The other end of the spring is connected to a rigid support as shown in Fig.7.11. Initially the block is at the mean position on a frictionless horizontal surface i.e. at rest and $x = 0$. When the block is displaced through a small distance 'x' to the right then according to Hook's law there is a restoring force which causes the oscillation of the mass spring system. The acceleration produced by restoring force is directed towards its mean position and is given as;

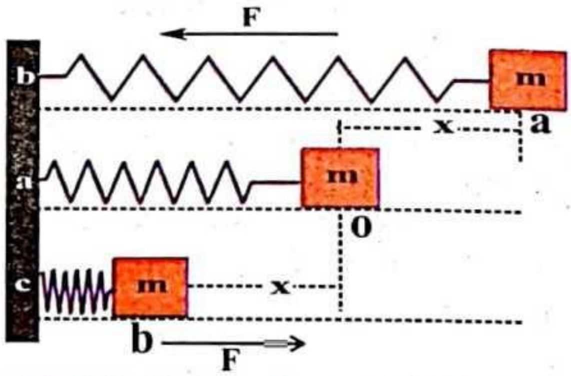


Fig.7.11: Horizontal mass spring system

$$a = -\left(\frac{k}{m}\right)x \quad \ldots\ldots(7.14)$$

Similarly, the acceleration of the particle moving in a circle executing simple harmonic motion is given as;

$$a = -\omega^2 x \quad \ldots\ldots(7.15)$$

Comparing equation (7.14) and equation (7.15)

$$\omega^2 = \frac{k}{m}$$

$$\omega = \sqrt{\frac{k}{m}} \quad \ldots\ldots(7.16)$$

Time Period

The time period of a mass attached to a spring, placed on a horizontal frictionless surface and executes S.H.M., is defined as time taken to complete its one round trip. Now,

$$T = \frac{2\pi}{\omega}$$

$$T = \frac{2\pi}{\sqrt{\frac{k}{m}}}$$

$$T = 2\pi\sqrt{\frac{m}{k}} \quad \ldots\ldots(7.17)$$

Displacement

The displacement 'x' of the mass attached to the spring at time 't' is given by;

$$x = r\cos\theta \qquad [\because \theta = \omega t]$$

$$x = r\cos\omega t$$

But in case of mass attached to a spring $r = x_o$ where x_o is its amplitude from mean position to extreme position as shown in Fig. 7.11.

Substitute the values of r and ω in $x = r\cos\omega t$

$$x = x_o \cos\sqrt{\frac{k}{m}}t \qquad \text{......(7.18)}$$

Instantaneous Velocity

We have studied that the velocity of the projection of the particle moving in a circle is along the horizontal direction and at any instant of time t is given by:

$$v = \omega\sqrt{r^2 - x^2}$$

But in mass spring system, we take $r = x_o$ and $\omega = \sqrt{\frac{k}{m}}$

$$v = \sqrt{\frac{k}{m}}\sqrt{x_o^2 - x^2}$$

CONCEPT CHECK

The period is the time required to complete one cycle.

$$v = \sqrt{\frac{k}{m}}\sqrt{x_o^2\left(1 - \frac{x^2}{x_o^2}\right)}$$

$$v = x_o\sqrt{\frac{k}{m}}\sqrt{\left(1 - \frac{x^2}{x_o^2}\right)} \qquad \text{......(7.19)}$$

At mean position $x = 0$ and $v = v_o$ (maximum) equation 7.19 becomes.

$$v = x_o\sqrt{\frac{k}{m}}\sqrt{\left(1 - \frac{0}{x_o^2}\right)}$$

$$v = x_o\sqrt{\frac{k}{m}}\sqrt{(1-0)}$$

$$v = x_o\sqrt{\frac{k}{m}} \qquad \text{......(7.20)}$$

Substitute equation (7.20) in equation (7.19)

$$v = v_o\sqrt{\left(1 - \frac{x^2}{x_o^2}\right)} \qquad \text{......(7.21)}$$

7.4 SIMPLE PENDULUM

A simple pendulum is an ideal pendulum which consists of a solid bob of mass 'm' suspended from a rigid support through a light inextensible string of length 'ℓ'. The pendulum stays at a fixed point if the string is in vertical position. This point is called mean or equilibrium position. The forces acting on the solid bob are,

(a) the weight of the pendulum '*mg*' acting downward and
(b) the tension 'T' of the string acting in the upward direction along the direction of string.

When the pendulum is displaced from its mean position O through an angle 'θ' to the extreme position 'P', then a restoring force acts on the pendulum towards the mean position. Due to this restoring force, the pendulum starts oscillation to and fro under the action of gravity along a curved path about the mean position 'O' as shown in Fig. 7.12. At extreme position 'P', the weight of the body makes an angle 'θ' with the direction of the string. We can resolve it into its rectangular components. As the pendulum has no motion along the direction of the string therefore, the component $mg\cos\theta$ and tension 'T' are along the same line but in opposite direction so they cancel the effects of each other The component mg $\sin\theta$ provides the necessary restoring force and is responsible for the motion of simple pendulum. Thus;

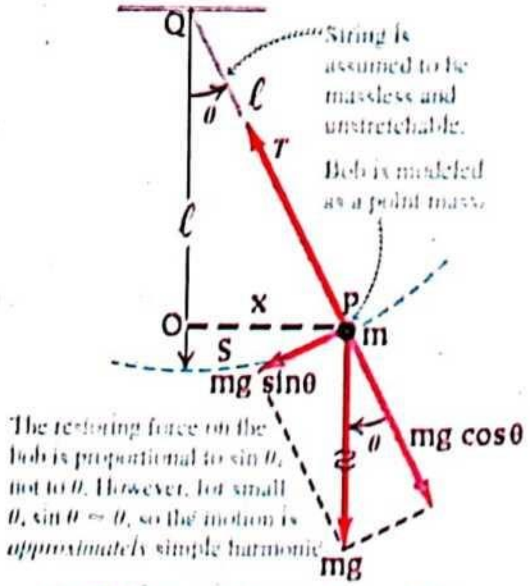


Fig.7.12: An ideal simple pendulum

$$F = -mg\sin\theta$$

Negative sign shows that the acceleration of the pendulum is always directed towards its mean position. According Newton's 2^{nd} law

$$F = ma$$

Comparing above equations

$$ma = -mg\sin\theta$$

$$a = -g\sin\theta \;......(7.22)$$

If the angle 'θ' of the simple pendulum is small (i.e., $\theta < 10^\circ$), then the $\sin\theta$ can be replaced by the angle θ itself, expressed in radians. That is, for small angles

$$\sin\theta \approx \theta.$$

So equation (7.22) is written as;

$$a = -g\theta \;......(7.23)$$

By the definition of an angular displacement,

$$\theta = \frac{S}{\ell}$$

where 'S' is the actual path length followed by the pendulum. Thus

$$a = -\left(\frac{g}{\ell}\right)S$$

In Figure 7.12, 'x' is very nearly equal to the arc of length 'S' of the circular path when the angle θ is small (about 10^{o} or less). Hence,

$$a = -\left(\frac{g}{\ell}\right)x \quad \ldots\ldots(7.24)$$

If length 'ℓ' of the pendulum is fixed and 'g' remains constant for a given place and (g/ℓ) is constant. Eq. (7.24) can be rewritten as;

$$a = -(\text{constant})\ x$$

$$a\ \alpha -x$$

This is the mathematical form of S.H.M and it is concluded that the motion of a simple pendulum is S.H.M.

As $\quad a = -\omega^2 x \quad \ldots\ldots(7.25)$

Comparing equation (7.24) and equation (7.25)

$$\omega^2 = \frac{g}{\ell}$$

$$\omega = \sqrt{\frac{g}{\ell}} \quad \ldots\ldots(7.26)$$

POINT TO PONDER

Does a vibrating simple pendulum produce any sound?

Time period of a simple pendulum is given as

$$T = \frac{2\pi}{\omega} \qquad \left(\because \omega = \frac{\theta}{t} = \frac{2\pi}{T}\right)$$

$$T = \frac{2\pi}{\sqrt{\frac{g}{\ell}}}$$

$$T = 2\pi\sqrt{\frac{\ell}{g}} \quad \ldots\ldots(7.27)$$

The above expression shows that the time period of simple pendulum is directly proportional to the square root of the length of the string and inversely proportional to the square root of acceleration due to gravity. The time period of motion of the pendulum is independent of the mass *m* of the bob and amplitude.

A pendulum that completes one vibration in two seconds, i.e., its time period is two seconds is known a second pendulum.

The simple pendulum can be used to determine the gravitational acceleration at a particular location. We measure the length l of the pendulum and then set the pendulum into motion. The time period T of the simple pendulum is measured using a stopwatch and the acceleration of gravity is calculated by using equation (7.27) in the following form;

$$g = 4\pi^2 \frac{\ell}{T^2} \quad \text{......(7.28)}$$

DO YOU KNOW

If a pendulum is shifted from Karachi to Quetta than its time period will be increased.

Example 7.4

What is the length of a second pendulum?

Solution:

$$\ell = ?$$

$$T = 2\ s$$

$$T = 2\pi\sqrt{\frac{\ell}{g}}$$

$$T^2 = \frac{4\pi^2\ell}{g}$$

$$\ell = \frac{gT^2}{4\pi^2} = \frac{(9.8)(2)^2}{4(3.14)^2} = 0.994\ \text{m}$$

$$\ell = 99.4\ \text{cm}$$

CONCEPT CHECK

A pendulum making small swings undergoes simple harmonic motion.

7.5 CONSERVATION OF ENERGY IN S.H.M

When a body is executing simple harmonic motion it possesses both potential energy as well as kinetic energy. Its potential energy is on account of its displacement from mean position and the kinetic energy is due to its velocity. These energies vary during the oscillation, but the total energy at any instant remains constant in the absence of unbalanced resistive forces. In case of mass-spring system, when the mass is displaced from the mean position 'O' then there is a restoring force (F) whose value is zero at mean position when $x = 0$ and its value is maximum at either extreme positions where $x = x_o$. Thus average value of force from the mean position to the extreme position is

$$F_{avg} = \frac{F_{mean} + F_{ext}}{2} = \frac{0 + F}{2} = \frac{F}{2}$$

When displacement = 0 Force = 0

When displacement = x_o Force = kx_o

$$F = \frac{1}{2} k x_o$$

When the spring is stretched to its maximum displacement x_o, work is done on the spring which is given as under;

$$W = \vec{F} \cdot \vec{d} = \frac{1}{2} k\, x_o \cdot x_o = \frac{1}{2} k\, x_o^2$$

This work done on the mass attached to a spring stores in terms of potential energy, called elastic potential energy. So we have

$$P.E = \frac{1}{2} k\, x^2 \quad ......(7.29)$$

It is clear from Eq. (7.29) that potential energy of simple pendulum is zero at $x = 0$ and maximum at $x = \pm x_o$ i.e. the extreme position on either side.

After the removal of force, the mass attached to a spring starts its motion with velocity v then the kinetic energy of the mass attached to spring is given as:

$$K.E = \frac{1}{2} m\, v^2$$

From eq. 7.19 $\quad v = x_o \sqrt{\frac{k}{m}} \sqrt{\left(1 - \frac{x^2}{x_o^2}\right)}$

$$K.E = \frac{1}{2} m \left(x_o \sqrt{\frac{k}{m}} \sqrt{1 - \frac{x^2}{x_o^2}} \right)^2$$

$$K.E = \frac{1}{2} k x_o^2 \left(1 - \frac{x^2}{x_o^2} \right) \quad ......(7.30)$$

We can study the values of P.E. and K.E. at different positions. Using Eqs.(7.29) and (7.30) respectively.

At mean position

At mean position where $x = 0$

Equation 7.29 and equation 7.30 becomes.

$$P.E. = \frac{1}{2} k\,(0)^2 = 0$$

$$K.E. = \frac{1}{2} k x_o^2 \left(1 - \frac{0}{x_o^2} \right)$$

$$K.E. = \frac{1}{2} k x_o^2$$

CONCEPT CHECK

The amplitude of vibrating body can be increased by the application of small forces at specific intervals.

$$T.E. = P.E + K.E$$

$$T.E. = 0 + \frac{1}{2} kx_o^2$$

$$T.E. = \frac{1}{2} kx_o^2 \quad ......(7.31)$$

We conclude that the potential energy of a simple pendulum, executing S.H.M., at mean position is zero and its kinetic energy is maximum.

At extreme position

At extreme position we have $x = \pm x_o$ and Eq.(7.30) becomes;

$$K.E = \frac{1}{2} kx_o^2 \left(1 - \frac{x_o^2}{x_o^2}\right) = \frac{1}{2} kx_o^2 (1-1) = \frac{1}{2} kx_o^2 (0)$$

$$K.E = 0$$

$$P.E = \frac{1}{2} kx_o^2$$

$$T.E = P.E + K.E$$

$$T.E = \frac{1}{2} kx_o^2 \quad ......(7.32)$$

We conclude that the kinetic energy of simple pendulum, executing S.H.M., at extreme positions on either side is zero and its potential energy is maximum.

At any position

At any position x, where $-x_o < x < x_o$ then we have,

$$K.E. = \frac{1}{2} kx_o^2 \left(1 - \frac{x^2}{x_o^2}\right)$$

$$P.E. = \frac{1}{2} k\,x^2$$

The total energy of simple pendulum, executing S.H.M., can be obtained by adding above two equations i.e.,

$$T.E. = P.E + K.E$$

$$T.E. = \frac{1}{2} kx^2 + \frac{1}{2} kx_o^2 \left(1 - \frac{x^2}{x_o^2}\right)$$

$$T.E. = \frac{1}{2} kx^2 + \frac{1}{2} kx_o^2 - \frac{1}{2} kx^2$$

$$T.E. = \frac{1}{2} kx_o^2 \quad ......(7.33)$$

Equations (7.31), (7.32) and (7.33) show that when a body executing SHM, the total energy of the vibrating system remains constant i.e., when the K.E. of the mass is maximum, mass passes through the centre of oscillation, the P.E of the mass spring is zero ($x = 0$). Conversely when the P.E. of the spring is maximum, mass is at its extreme position on either side, the K.E. of the mass is zero ($v = 0$). Fig 7.13 shows the variation of P.E. and K.E. with displacement 'x'. But the total energy (T.E.) of the vibrating system remains constant and this is represented by the horizontal line (brown line).

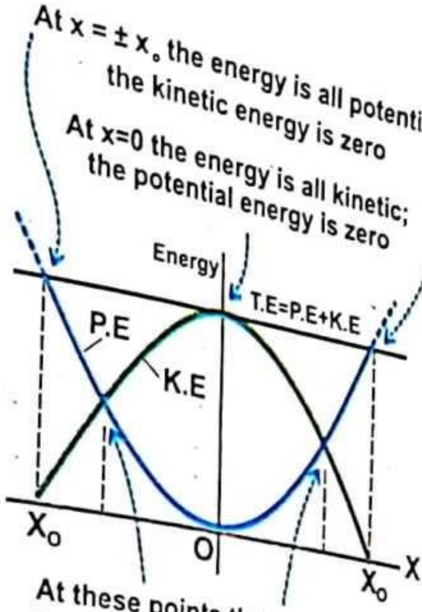


Fig.7.13: Potential energy, Kinetic energy and Total energy for linear harmonic oscillator.

7.6 FREE OSCILLATION

Consider a body or a system capable of oscillating, which is displaced from its mean position to its extreme position and then left free. Due to the restoring force, it starts oscillation with certain frequency which is called its natural frequency and the corresponding period is called its natural time period. If a body is oscillating with its own natural frequency and it is free from all the external resistive forces then such oscillations of the body are called free oscillations.

For example, oscillations of a simple pendulum, vibrations of prongs of a tuning fork, vibrations of string of musical instrument etc.

In free oscillations, the total energy of the body remains constant i.e. it is conserved. As we are assuming the absence of resistance force therefore the amplitude of the oscillation remains constant. Graphically, the free oscillations of a body with constant amplitude are shown in Fig.7.14.

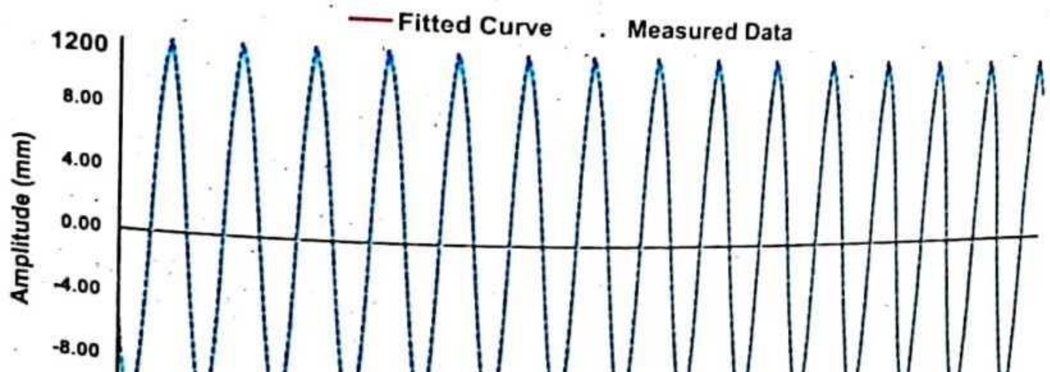


Fig.7.14: A graph of free oscillation of a body with constant amplitude

7.7 DAMPED OSCILLATION

In free oscillation, we have studied that the total mechanical energy of the oscillating body remains constant. But in practices, when a body is oscillating with its natural frequency, the amplitude of the oscillation gradually decreases with time and finally it comes to rest. This is due to the presence of resistive forces such as; air resistance, friction etc. The oscillation with decreasing amplitude in the presence of various resistive forces is called damped oscillation and the resistive forces are called damping forces. Energy dissipates due to negative work done by these damping forces and the body comes to rest in due course of time.

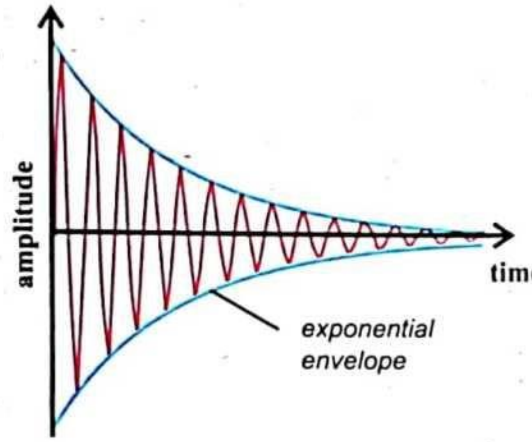


Fig.7.15: Damped oscillation of a body with decreasing amplitude

The damping force depends upon the speed of the oscillating body and is directed opposite to the velocity. Graphically the damped oscillation of the oscillating body is shown in Fig. 7.15.

Now the damped oscillation can be studied under the following three different cases.

(i) When the damping force is greater than the oscillating force, the body does not oscillate, i.e., without performing any oscillation, the body quickly comes at rest position. Such motion is called over-damping; graphically the over damping of a body is shown in Fig. 7.16.

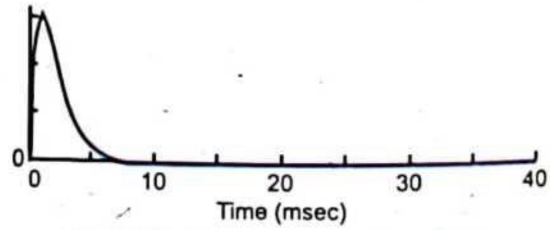


Fig.7.16: Over damping by a body

(ii) When the damping force is equal to the oscillating force, then the motion of body is called critical damping. In this case, the body returns to the equilibrium (mean) position with uniform speed along a curved path without performing oscillation as shown in Fig. 7.17.

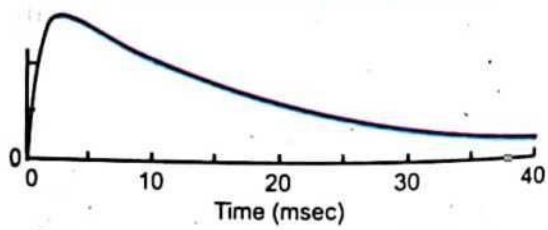


Fig.7.17: Critical damping of a body

(iii) When the damping force is less than the oscillating force then the body is set into oscillation and is called under-damping. Graphically, the under-damping of a body is shown in Fig. 7.18.

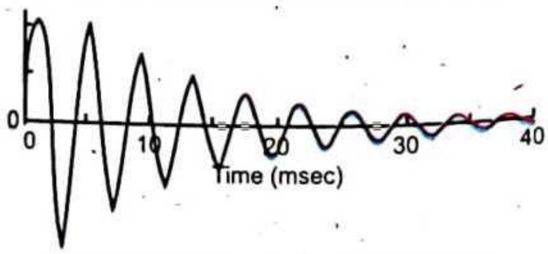


Fig.7.18: Under damping of a body

Examples of damping devices

Shock absorber

In damped oscillation, a small fraction of the energy of the oscillating system is dissipated against the friction but damping in some cases is very useful. One widely used application of damped oscillation is in the suspension system of an automobile. A shock absorber is attached to the frame of the vehicle.

A shock absorber is designed to use damping forces, which reduce the vibrations related with a bumpy ride.

As Fig.7.19 shows, a shock absorber consists of a piston in a reservoir of viscous fluid such as oil. When the piston moves up and down in response to a bump on the road, the oil inside the pressure tube is forced to go through piston valve and the base valve to move into the adjacent chamber. The holes in the valve control the rate of oil flow. Viscous forces that arise during this movement cause the damping effect.

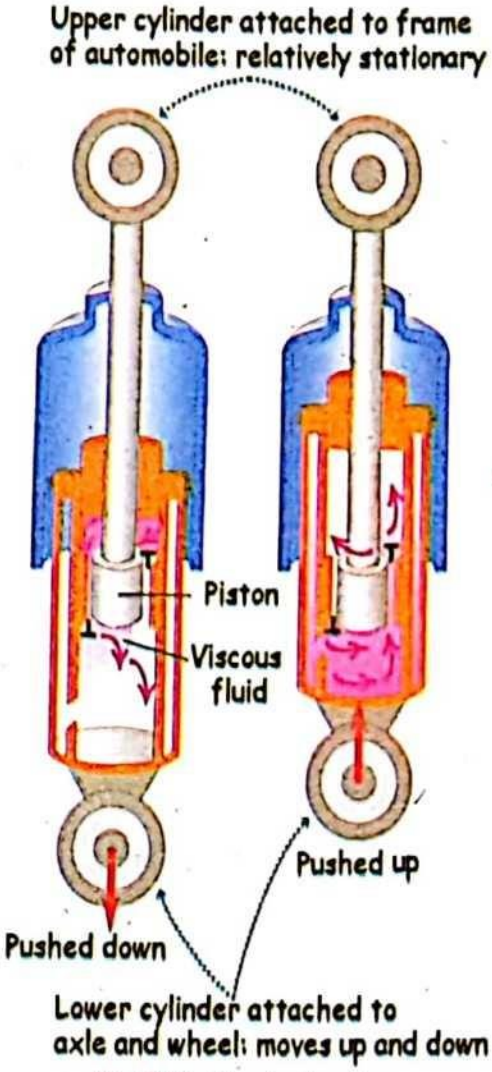


Fig.7.19: Shock absorber

The idea behind a shock absorber is to ease the natural bouncing motion of a spring. The degree of damping of shock absorber is shown in Fig.7.20. If the shocks are worn, and the system becomes under damped motion, then that wheel is going to be bouncing down the road (red-line). If the shocks are too aggressive, then it can create a situation where it delays the time it takes for the tyre to rebound to its position before the bump (green line).

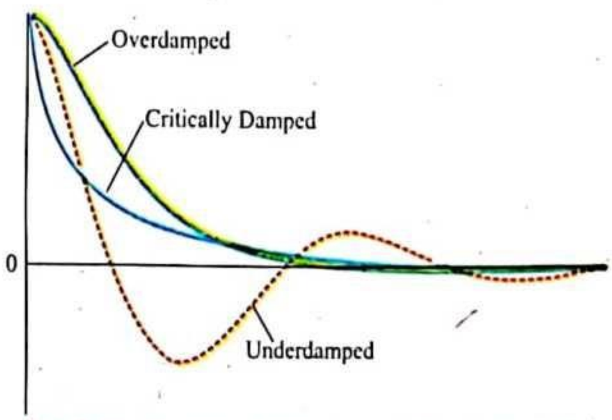


Fig.7.20: Degree of damping of Shock absorber

At critical damping, the tyre will rebound as quickly as it can to the road, without overshoot (blue line). In reality, critical dampening does not occur rather it slightly turns under-damped for a more comfortable ride. Typical automobile shock absorbers are designed to produce underdamped motion somewhat like that red line.

7.8 FORCED OSCILLATION AND RESONANCE

In damped oscillation, the oscillator cannot maintain its natural frequency for long duration due to the resistive forces and the amplitude of the oscillation decreases gradually with time. But we can maintain constant amplitude by applying a periodic external force which is called a driving force. Thus when the oscillating body is subjected to a periodic driving force then such oscillation is called forced oscillation and its frequency is called driving frequency. The vibration of a vehicle caused by the running of engine is an example of forced vibration. In forced oscillation, the amplitude of the oscillation depends upon the relation between the driving frequency and the natural frequency of the body.

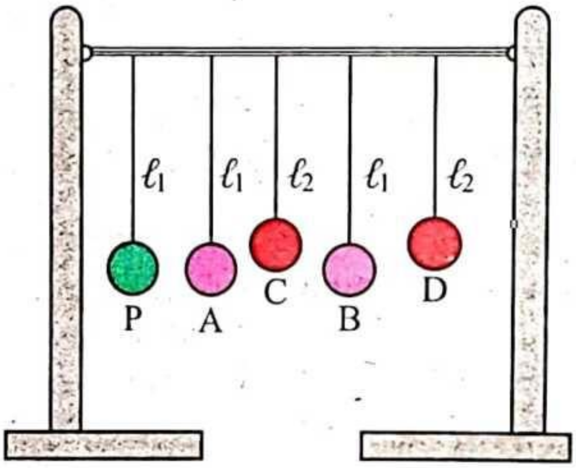


Fig.7.21: A set of five simple pendulums of different lengths suspended from a common rod.

If the frequency of the driving force is same as the natural frequency of the oscillating body, the amplitude of vibration is very much increased. This phenomenon is known as resonance and the oscillations of large amplitude are called resonant oscillations.

To demonstrate the resonance phenomenon, we perform a simple experiment. The experimental set up consists of two pair of pendulums A & B and C & D such that the length of A and B is ℓ_1 and length of C and D is ℓ_2. All the pendulums are suspended by a horizontal rod as shown in Fig. 7.21.

Now we introduce another pendulum 'P' whose length can be varied i.e. either ℓ_1 or ℓ_2. Consider the case when the length of pendulum 'P' is equal to ℓ_1. If the pendulum 'P' is set into vibration, this vibration reaches the other pendulums through the rod. Then the pendulums A and B receive a driving force through the rod and they also start vibration and its amplitude increases due to the resonance phenomenon because their lengths, natural frequency and natural periods are same. At the same time the pendulums C and D whose natural frequencies are different from natural frequency of 'P' do not oscillate i.e. they continue to remain at rest. If the length of the pendulum 'P' is made equal to ℓ_2 and allowed to vibrate, then the pendulums C and D start vibration due to resonance while pendulum 'A and B' remain at rest.

The resonance phenomenon can further be explained by some examples;

(i) The soldiers are advised to break their steps while crossing a bridge. If the soldiers march in steps then it is possible that the frequency of their footsteps become equal to the natural frequency of the bridge and the bridge may be set into vibrations with large amplitude due to the resonance.

(ii) During earthquake, when the frequency of earthquake is equal to the natural frequency of a building then the building will be set into vibrations with large amplitude due to the resonance and the building may collapse.

(iii) In communication system, all the transmitting signals can be received by receivers due to the resonance phenomenon when the frequency of the receiver is made equal to the frequency of incoming signal.

(iv) Microwave ovens generate super high frequency electromagnetic waves (3GHz-30GHz and wavelength of about 12 cm) and scatter them throughout the oven. The frequency of microwave excites water molecules into resonance and causes them to collide with one another. Friction generated by the collisions changes the kinetic energy of the water into heat that warms the food. Food containing water molecules can only be heated by the microwave oven.

(v) The amplitude of a swing can be increased by applying a suitable periodic force on it.

(vi) The tuning of a radio set for a certain station is also based on resonance in its LC-circuit.

7.9 SHARPNESS OF RESONANCE

We have studied in the resonance phenomenon that the amplitude of the oscillation is maximum when the frequency of the driving force is nearly equal to the natural frequency of the oscillating body. The amplitude can be decreased by changing the frequency of driving force.

If the amplitude of oscillation increases rapidly at a frequency 'f' slightly different that from the resonant frequency 'f_o', then the resonance is said to be sharp. Amplitude of the resonance oscillation and its sharpness depend upon damping that is, smaller the damping, greater will be the amplitude and more sharp will be the resonance. Similarly, for greater damping, the amplitude of the resonant oscillation will be small and such resonance is called flat resonance. Fig.7.22 shows the

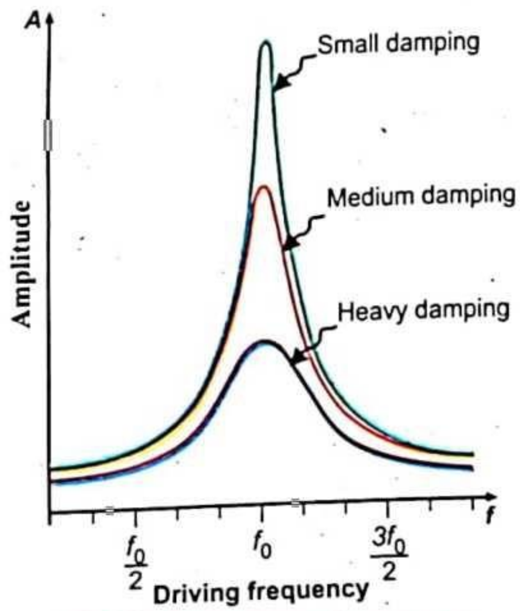


Fig.7.22: Sharpness of resonance

amplitude as a function of the applied frequency of the driving force. We see that the amplitude is large if the damping is small. Also the resonance is sharp in this case, that is the amplitude rapidly falls if 'f' is different from 'f_o'.

In the absence of damping forces, the amplitude of the oscillation (forced vibration) will be infinity but practically it is impossible. In all real cases some damping is always present in mechanical systems and the amplitude remains finite.

However, the amplitude may become very large if the damping is small and the applied frequency is close to the natural frequency.

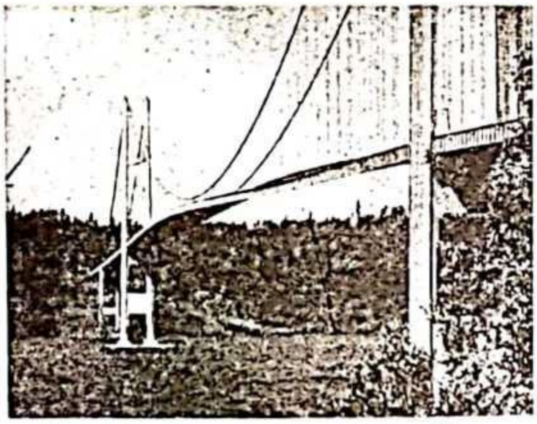


Fig.7.23: Before resonance condition Tacoma narrow bridge

The resonance effect is very important in the design of bridges and other civil engineering projects. On July, 1940 the newly constructed Tacoma Narrow Bridge (Washington) was opened for traffic as shown in Fig.7.23. Only four months after this, a mild wind set up the bridge in resonant vibrations. In a few hours the amplitude became so large that the bridge could not stand the stress and a part broke off and went into the water below (Fig.7.24). After this incident the engineers considered the resonance phenomenon in the design and construction of long span bridges.

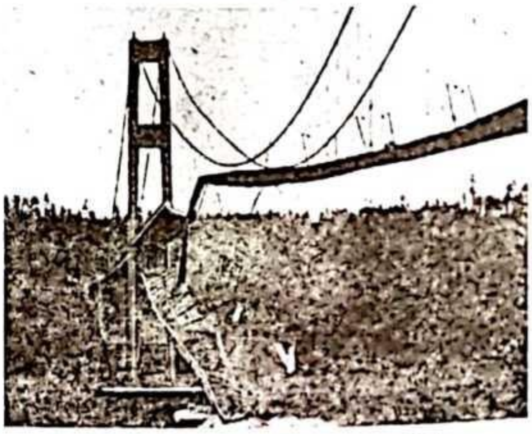


Fig.7.24: After resonance condition Tacoma narrow bridge

SUMMARY

- **Oscillatory motion:** To and fro motion of a body about its mean position is called oscillatory motion.
- **Periodic motion:** Motion that repeats itself in equal intervals of time.
- **Displacement:** The distance of a vibrating body from its equilibrium position to its present position.
- **Simple Harmonic Motion:** The motion of a body is said to be S.H.M. if its acceleration is directly proportional to the displacement and is always directed towards the mean position.
- **Vibration/Cycle:** The complete round trip of an oscillating body is called vibration.

- **Amplitude:** The maximum displacement from mean position of an oscillating object is called its amplitude.
- **Time Period:** The time taken by the vibrating body to complete one vibration / cycle.
- **Frequency:** Number of vibrations in one second is called frequency.
- **Circular Motion related with SHM:** The movement of projection of particle moving in a circle is S.H.M.
- **Angular Frequency:** The number of vibrations per unit time is called angular frequency.
- **Simple Pendulum:** A simple pendulum consists of a solid bob suspended by a string from a rigid support. Its to and fro motion about its mean position is S.H.M and its time period depends upon its length i.e. $T = 2\pi\sqrt{\frac{\ell}{g}}$.
- **Inter conversion of energy in SHM:** When a body is executing S.H.M. then it possesses both K.E. and P.E. which are inter-convertible such that the total energy remains constant.
- **Free Oscillations:** The oscillation of a body in the absence of resistive force is called free oscillation.
- **Forced oscillations:** The oscillation which is driven by frequency of a periodic force is known as forced oscillation.
- **Damped Oscillations:** The oscillation of a body in a resistive medium with decreasing amplitude is known as damped oscillation.
- **Over damped:** When the damping force is greater than the oscillating force then it is called over damping.
- **Critical damping:** When the damping force is equal to the oscillating force then the motion of the body is called critical damping.
- **Under damping:** When damping force is smaller than the oscillating force then the motion of the body is called under-damping.
- **Natural time period and natural frequency:** In the absence of resistive forces, the time period and frequency of the oscillating body is called its natural period and natural frequency.
- **Resonance:** When a force is applied, whose frequency is equal to the natural frequency of the system, the system vibrates at maximum amplitude and the phenomenon is called resonance.

- **Sharpness of resonance:** When the frequency of driving force is slightly different from the resonance frequency then the amplitude will be increased and resonance will be sharp. Sharpness of resonance depends upon damping.

EXERCISE

Multiple choice questions.

1. The acceleration of a body executing simple harmonic motion is;
   (a) Zero at each point (b) Remain same at each point
   (c) Maximum at mean position (d) Minimum at extreme position
2. What is the value of a spring constant when a 100g mass is attached to a spring and it is accelerated $0.5 m\ s^{-2}$ through a displacement of 5cm?
   (a) $0.1\ N\ m^{-1}$ (b) $0.5\ N\ m^{-1}$ (c) $1\ N\ m^{-1}$ (d) $5\ N\ m^{-1}$
3. If a spring of forced constant K is cut into two equal parts, then the spring constant of each half is
   (a) $\frac{K}{2}$ (b) 2K (c) K (d) $\frac{K}{\sqrt{2}}$
4. When a body is performing S.H.M then at its extreme position.
   (a) Displacement is zero (b) Amplitude is zero
   (c) Velocity is a zero (d) P.E is zero
5. A particle is executing S.H.M along a straight line with amplitude A, its kinetic energy is maximum when its displacement is
   (a) $\pm A$ (b) $\pm\frac{A}{2}$ (c) zero (d) $\pm\frac{A}{\sqrt{2}}$
6. The time period of a body attached to a spring depends upon.
   (a) Amplitude (b) Mass
   (c) Length (d) Displacement
7. When the length of the pendulum is increased four times then its time period is increased.
   (a) One time (b) Two time (c) Three time (d) Four time
8. What is the frequency of the body when its time period is 2 seconds?
   (a) 1 Hz (b) 2 Hz (c) 0.2 Hz (d) 0.5 Hz
9. A second's pendulum is one who has a time period of
   (a) 1 s (b) 2 s (c) $\frac{1}{2}$s (d) 0.2 s
10. In S.H.M., at what distance from mean position in terms of amplitude x_o, K.E. and P.E. both will have equal value?

(a) $0.51x_o$ (b) $0.61x_o$ (c) $0.71x_o$ (d) $0.81x_o$

11. The instantaneous K.E of a mass attached to the end of an elastic spring is:

(a) $\frac{1}{2}k(x_o^2 - x^2)$ (b) $\frac{1}{2}k(x_o^2 + x^2)$ (c) $\frac{1}{2}k(x^2 - x_o^2)$ (d) $\frac{1}{2}k(x - x_o)$

12. In S.H.M., we have the conservation of

(a) Kinetic energy (b) Potential energy
(c) Total energy (d) Mechanical energy

13. A free oscillation has constant

(a) Energy (b) Amplitude (c) Frequency (d) All of these

14. When damping force is equal to the oscillating force then the damping is called.

(a) Under damping (b) Critical damping
(c) Over damping (d) No damping

15. A resonance occurs when driving frequency is

(a) Less than the natural frequency (b) Greater than natural frequency
(c) Equal to the natural frequency (d) Equal to zero

16. In the absence of damping force, when driving frequency is equal to the oscillating frequency then the amplitude of the oscillation will become

(a) Zero (b) Minimum (c) Maximum (d) Infinity

17. Food is cooked in a microwave oven by the effect of:

(a) Interference (b) Mechanical resonance
(c) Magnetic resonance (d) Electric resonance

SHORT QUESTIONS

1. What is the role of the restoring force in the simple harmonic motion?
2. What are the two main conditions that must be met to produce simple harmonic motion?
3. What is the relation of circular motion of body with simple harmonic motion?
4. What is the difference between the time periods of simple pendulum and a body attached to a spring?
5. What is the difference between phase angle and phase constant?
6. When a body is performing S.H.M then at what condition, its total mechanical energy is conserved and at what condition, its energy does not conserve?
7. Show that in S.H.M. the acceleration is zero when the velocity is maximum and the velocity is zero when the acceleration is maximum.
8. In the simple harmonic motion of a mass attached to a spring, the velocity of the mass is equal to zero when the acceleration has its maximum value. How is

this possible? Can you think of other examples in which a body has zero velocity with a nonzero acceleration?

9. What is the difference between free and forced oscillation?
10. Give one practical example each of free and forced oscillations.
11. How natural time period of an oscillating body remains constant.
12. Describe the three kinds of damping?
13. How does sharpness of resonance occur?
14. How the amplitude of resonant oscillation affected by damping?
15. What happens to the time period of a simple pendulum if its length is quadrupled?

COMPREHENSIVE QUESTIONS

1. Define simple harmonic motion with all its characteristics such as; Vibration, Instantaneous displacement, Amplitude, Time period, Frequency and Angular.
2. Show that if a particle is moving along a circle, then its projection on the diameter of the circle executes S.H.M.
3. Prove that the motion of a mass attached to a spring is executing S.H.M.
4. Describe simple pendulum and prove that its time period depends upon its length.
5. Prove that when a body is performing S.H.M, its total energy remains constant.
6. Compare free and damped oscillations. Also discuss the three types of damped oscillations.
7. State and explain with examples the forced oscillation and resonance.

NUMERICAL PROBLEMS

1. When a 600 g mass is suspended at the end of a vertical spring then the spring stretches by 0.45 m. What is the spring constant of the spring and how much further will it be stretched if an additional mass of 600 g is hung from it?
   ($13\ Nm^{-1}$, 0.45 m)
2. A 2 kg mass attached to a spring is executing S.H.M. and makes 4 vibrations per second. Calculate the acceleration and the restoring force acting on the body when its displacement from mean position is 7 cm. **($44.2\ m\ s^{-2}$, 88.4 N)**
3. A particle performing S.H.M of amplitude 8 cm. If its velocity while crossing the mean position is $4\ m\ s^{-1}$, what is its frequency and time period?
   (8 Hz, 0.125 sec)
4. What is the amplitude, frequency, period and position at t = 2s of a vibrating body whose motion is represented by the equation $x = 0.2\cos 0.125\pi t$?
   (0.2 m, 0.0625 Hz, 16 s, 0.20 m)

5. Calculate the frequency of simple pendulum of length 0.8 m which is vibrating on Mars, where weight of object is 0.40 times its weight on earth. **(0.35 Hz)**

6. How much time period of a simple pendulum is increased by increasing its length from 0.8 m to 0.993 m? **(0.2 s)**

7. A block of mass 5 kg is dropped from a height of 0.8 m on to a spring of spring constant 1960 N m^{-1}. Find the displacement through which the spring will be compressed. **(0.2 m)**

8. A car of mass 1300 kg is constructed using a frame supported by four springs. Each spring has a spring constant 20000 N m^{-1}. If two people riding in the car have a combined mass of 160 kg, find the frequency of vibration of the car when it is driven over a pot-hole in the road. Assume the weight is evenly disturbed. **(1.18 Hz)**