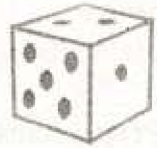
6 Probability



$0 \leq P(A) \leq 1$

6.1 Introduction

If you bought seven tickets for a raffle out of 700 tickets sold altogether. Each of 700 tickets is as likely as any other to be drawn for first prize, you would say that you had 7 chances out of 700, or a single chance out of 100 for winning the first price.

Probability gives us a measure for the **likelihood** that some thing will happen. However, probability cannot predict the number of times that an occurrance actually happens. Most of the decisions that affect our daily lives are based upon likely hood and not on absolute certainty.

In this chapter, we shall develop methods to deal with problems concerned with chance events.

Some definitions, terminologies and notations are explained below to enable the student to certain categories of situation precisely and move briefly.

Sets: A set is a well defined collection of distinct objects. The objects making up a set are called its elements. A set is usually denoted by a capital letter i.e., *A, B, C* etc. while its elements are denoted by small letters i.e., *a, b, c* etc. For example, the set *A* that consists of first five positive integers can be described as:

$$A = \{1, 2, 3, 4, 5\}$$

Here, for 3 belongs to set *A*, we write $3 \in A$; and read it as 3 belongs to A, while for 6 does not belong to set *A*, we write $6 \notin A$ and read it as 6 does not belong to *A*.

Null Set: A set that contains no element is called null set or empty set. It is denoted by { } or Φ.

Subset: If every element of a set A is also an element of a set B, A is said to be a subset of B and it is denoted by; $A \subseteq B$

Proper Subset: If A is a subset of B, and B contains at-least one element which is not an element of A, A is said to be a proper subset of B and is denoted by;

$$A \subset B$$

Finite and Infinite Sets: A set is finite, if it contains a specific number of elements, i.e., while counting the members of the set, the counting process comes to an end otherwise the set is an infinite set. For example; $A = \{1, 2, 3, 5\}$, $B = \{x, y, z, t, u\}$ and $C = \{x \mid x \text{ is month of years}\}$ are finite sets.

Whereas $D = \{2, 4, 6, 8, \ldots\}$ and $E = \{y \mid y \text{ is a point on a line}\}$ are infinite sets.

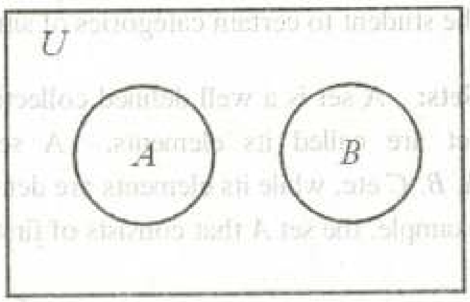
Universal Set: A set consisting of all the elements of the sets under consideration is called the universal set. It is denoted by U.

For example, if $A = \{1, 2, 3\}$, $B = \{2, 4, 5, 6\}$, $C = \{8, 10\}$, then $U = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}$,

Disjoint Set: Two sets A and B are said to be disjoint sets, if they have no elements in common i.e., if $A \cap B = \Phi$, A and B are disjoint sets.

$A = \{6, 8, 10, 12\}$ and $B = \{1, 4, 9, 11\}$

are disjoint sets as $A \cap B = \Phi$



$A \cap B = \Phi$

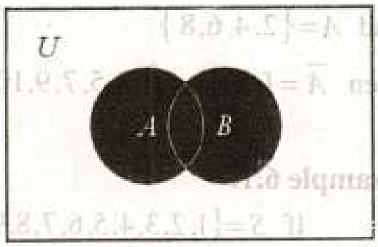
Overlapping Sets: Two sets A and B are said to be overlapping sets, if they have at-least one element in common, i.e., if $A \cap B \neq \Phi$ and none of them is the subset of the other set then A and B are overlapping sets.

For example $A = \{1, 3, 5, 7\}$ and $B = \{1, 4, 8\}$ are overlapping sets, as $A \cap B = \{1\} \neq \Phi$ and none of A and B is the subset of the other.

Venn Diagram:

Venn diagram is a diagram in which universal set U is represented by a rectangle and its subset is represented by a circle. In other words Venn diagram represents the relationship between sets by means of diagram.

Union of Sets: Union of two sets A and B is a set that contains the elements either belonging to A or to B or to both. It is denoted by $A \cup B$ and read as A union B. For example, if $A=\{1,2,3,4,5\}$ and $B=\{2,4,6,8,10\}$ then $A \cup B=\{1,2,3,4,5,6,8,10\}$

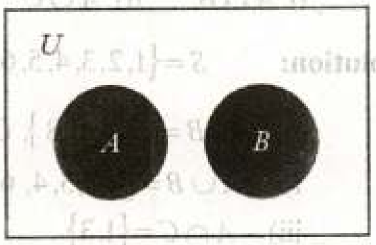


A ∪ B is shaded area

Let: $A=\{2, 4, 6\}$

$B=\{1, 3, 5\}$

then $A \cup B=\{1, 2, 3, 4,5, 6\}$

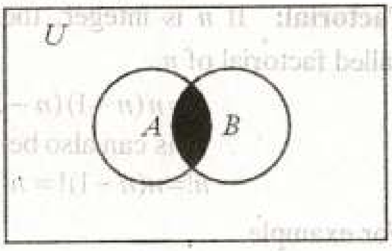


A ∪ B is shaded area

Intersection of Sets: Intersection of two sets A and B is a set that contains the elements belonging to both A and B. It is denoted by $A \cap B$ and read as A intersection B. For example, if

$A=\{1,2,3,4,5,6\}$ and $B=\{2,3,6,7\}$

then $A \cap B=\{2,3,6\}$



A ∩ B is shaded area

Difference of Sets: The difference of a set A and a set B is the set that contains the elements of the set A which are not contained in B. The difference of sets A and B is denoted by $A-B$. For example,

if $A=\{1,2,3,4,5,6,7,8\}$ and $B=\{2,4,6,8\}$

then $A-B=\{1,3,5,7\}$

A − B is shaded area

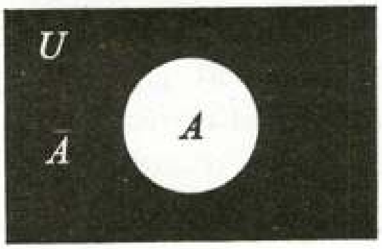
Complement of a Set: Complement of a set A denoted by $\overline{A}$ or A^c, is defined as $\overline{A} = U - A$.

For example if

$U = \{1,2,3,4,5,6,7,8,9,10\}$

and $A = \{2,4,6,8\}$

then $\overline{A} = U - A = \{1,3,5,7,9,10\}$



$\overline{A}$ is shaded area

Example 6.1:

If $S=\{1,2,3,4,5,6,7,8,9,10\}$, $A=\{1,2,3,4\}$ $B=\{2,4,6,8\}$, $C=\{1,3,5,7\}$ and $D=\{2,4\}$, then find :

i) $A \cup B$ ii) $A \cup C$ iii) $A \cap C$ iv) $C \cap B$ v) $\overline{C}$ vi) $\overline{A}$

Solution: $S=\{1,2,3,4,5,6,7,8,9,10\}$, $A=\{1,2,3,4\}$

$B=\{2,4,6,8\}$, $C=\{1,3,5,7\}$ and $D=\{2,4\}$

i) $A \cup B=\{1,2,3,4,6,8\}$ ii) $A \cup C=\{1,2,3,4,5,7\}$

iii) $A \cap C=\{1,3\}$ iv) $C \cap B = \Phi$

v) $\overline{C}=\{2,4,6,8,9,10\}$ vi) $\overline{A}=\{5,6,7,8,9,10\}$

Factorial: If n is integer, the product of first n positive integers denoted by $n!$ is called factorial of n.

$n! = n(n-1)(n-2)$ 3.2.1

This can also be written as :

$n! = n(n-1)! = n(n-1)(n-2)!$

For example,

$2! = 2 \times 1 = 2$

$4! = 4 \times 3 \times 2 \times 1 = 24$

$10! = 10 \times 9 \times 8 \times 7 \times 6 \times 5 \times 4 \times 3 \times 2 \times 1 = 3628800$

6.2 Permutations

An arrangement of finite number of objects in a definite order is called permutation of these objects. The number of ways of arranging n objects taken r at a time is denoted by ^nP_r and is defined as:

$${}^{n}P_{r} = \frac{n!}{(n-r)!}$$

The number of permutation of n objects out of which n_1 are alike of one kind, n_2 are alike of second kind and so on, n_k are alike of kth kind is given by

$$\binom{n}{n_1, n_2, \dots n_k} = \frac{n!}{n_1!\, n_2!\, n_3! \dots\, n_k!}$$

Example 6.2: How many distinct four–digit numbers can be formed from the following integers 1, 2, 3, 4, 5, 6 if each integer is used only once?

Solution: $\because {}^{n}P_{r} = \frac{n!}{(n-r)!}$

Here, $n = 6$ and $r = 4$

$$\therefore {}^{6}P_{4} = \frac{6!}{(6-4)!} = \frac{6!}{2!} = \frac{6\times5\times4\times3\times2\times1}{2\times1} = 360$$

Example 6.3: Evaluate: i) ${}^{5}P_{3}$ ii) ${}^{10}P_{6}$

Solution:

i) $${}^{5}P_{3} = \frac{5!}{(5-3)!} = \frac{5!}{2!} = \frac{5\times4\times3\times2\times1}{2\times1} = 60$$

ii) $${}^{10}P_{6} = \frac{10!}{(10-6)!} = \frac{10!}{4!} = \frac{10\times9\times8\times7\times6\times5\times4\times3\times2\times1}{4\times3\times2\times1} = 151200$$

6.3 Combinations

When a selection of objects is made without paying regard to the order of selection, it is called combination. The number of combinations of n things taken r at a time is denoted by ${}^{n}C_{r}$ or by $\binom{n}{r}$ and is defined by

$$\binom{n}{r} = \frac{n!}{r!(n-r)!}$$

Example 6.4: Evaluate: (i) ${}^5C_3$ (ii) ${}^6C_2$ (iii) ${}^8C_5$

Solution:

i) $${}^5C_3 = \frac{5!}{3!(5-3)!} = \frac{5!}{3!\ \ 2!} = 10$$

ii) $${}^6C_2 = \frac{6!}{2!(6-2)!} = \frac{6!}{2!\ \ 4!} = 15$$

iii) $${}^8C_5 = \frac{8!}{5!(8-5)!} = \frac{8!}{5!\ \ 3!} = 56$$

Random Experiment: Random experiment is an experiment which produces different outcomes even if it is repeated a large number of times under similar conditions. A random experiment has the following properties:

i) The experiment can be repeated any number of times.

ii) A random trial consists of at least two possible outcomes.

Sample Space: A set representing all possible outcomes of a random experiment is called sample space. It is denoted by *S*. Each element in a sample space is called sample point. For example, when a coin is tossed, the sample space is given by

$$S = \{H, T\}$$

If a coin is tossed two times, the sample space is given by

$$S = \{HH, HT, TH, TT\}$$

In throwing a die, the sample space is given by

$$S = \{1,2,3,4,5,6\}$$

When two dice are thrown, the sample space is given by

$$\begin{aligned} S = \{&(1,1), (1,2), (1,3), (1,4), (1,5), (1,6), \\ &(2,1), (2,2), (2,3), (2,4), (2,5), (2,6), \\ &(3,1), (3,2), (3,3), (3,4), (3,5), (3,6), \\ &(4,1), (4,2), (4,3), (4,4), (4,5), (4,6), \\ &(5,1), (5,2), (5,3), (5,4), (5,5), (5,6), \\ &(6,1), (6,2), (6,3), (6,4), (6,5), (6,6)\} \end{aligned}$$

Event: Any subset of the sample space is called an event. In a sample space there can be two or more events consisting of sample points. For example, when a fair dice is rolled, the coming up of an even number upward is an event i.e., {2, 4, 6} is an event. Similarly, the coming up of odd numbers is an event i.e., {1, 3, 5} is an event.

Simple Event: If an event consists of one sample points, it is called simple event. For example, when two coins are tossed, the event {TT} that two tails appear is a simple event.

Compound Event: If an event consists of more than one sample points, it is called a compound event. For example, when two dice are rolled, an event B, the sum of two faces is 4 i.e., $B = \{(1, 3), (2, 2), (3, 1)\}$ is a compound event.

Independent Events: Two events A and B are said to be independent, if the occurrence of one does not affect the occurrence of the other. For example, in tossing two coins, the occurrence of a head on one coin does not affect in any way the occurrence of a head or tail on the other coin.

Dependent Events: Two events A and B are said to be dependent, if the occurrence of one event affects the occurrence of the other event.

Mutually Exclusive Events: Two events A and B are said to be mutually exclusive, if they cannot occur together i.e., $A \cap B = \Phi$

In other words, the two events are called mutually exclusive events, if they are disjoint. For example, in toss of a coin, either the head or the tail will appear, but they cannot appear together. The appearance of head and the appearance of tail are mutually exclusive.

Equally Likely Events: Two events are said to be equally likely, if they have the same chance of occurrence. For example, when a coin is tossed, it is just as likely to occur heads as to occur tails.

Exhaustive Events: When a sample space S is partitioned into some mutually exclusive events, such that their union is the sample space itself, the event are called exhaustive event. Let a die is rolled, the sample space is given by

$S = \{1, 2, 3, 4, 5, 6\}$

Let $A = \{1, 2\}$, $B = \{3, 4, 5\}$, $C = \{6\}$

A, B and C are mutually exclusive events and their union $A \cup B \cup C = S$ is the sample space, so the events A, B and C are exhaustive.

6.4 Probability

Classical or "A prior" definition: If there are n equally likely, mutually exclusive and exhaustive outcomes and m of which are favourable to the occurrence of an event A then the probability of the occurrence of the event A, denoted by $P(A)$ is defined by the ratio m/n i.e.,

$$P(A) = \frac{\text{no.of favourable outcomes}}{\text{no.of possible outcomes}} = \frac{m}{n}$$

Relative frequency or a posteriori definition: If an experiment is repeated a large number of times say n under uniform conditions and if the event A occurs m times, then the probability of the occurrence of the event A is defined by the relative frequency m/n which approaches a limits as n increases i.e.,

$$P(A) = \lim_{n \to \infty} \frac{m}{n}$$

Mathematical Definition: The probability that an event A will occur, is the ratio of the number of sample points in A to the total number of sample points in S. i.e.,

$$P(A) = \frac{\text{no. of sample points in } A}{\text{no. of sample points in } S} = \frac{n(A)}{n(S)}$$

$P(A)$ must satisfy the following axioms:

i) $P(A) \geq 0$, which means that, probability of an event cannot be negative.

ii) $0 \leq P(A) \leq 1$ i.e., Probability of an event lies between 0 and 1

iii) $P(S) = 1$, which means that, sum of the probabilities is equal to one.

iv) If A and B are two mutually exclusive events, then

$$P(A \cup B) = P(A) + P(B)$$

Examples 6.5: A student solved 128 questions from first 200 questions of a book to be solved. What is the probability that he will solve the remaining all questions?

Solution: $n = 200$, $m' = 128$, $m = n - m' = 200 - 128 = 72$

$$\because P(A) = \frac{m}{n}$$

$$\therefore P(A) = \frac{72}{200} = 0.36$$

Example 6.6: A bag contains 4 red and 6 green balls out of which 3 balls are drawn. Find the probability of drawing

i) 2 red and 1 green balls. ii) all red balls.
iii) one green ball. iv) no red ball.

Solution:

Red	Green	Total
4	6	10

Balls to be drawn = 3

$$\text{Sample space} = \binom{10}{3} = 120$$

i) Let A be the event of drawing 2 red and one green ball

$$n(A) = \binom{4}{2}\binom{6}{1} = 36$$

$$P(A) = \frac{n(A)}{n(S)} = \frac{36}{120} = 0.30$$

ii) Let B be the event of drawing all red balls.

$$n(B) = \binom{4}{3}\binom{6}{0} = 4$$

$$P(B) = \frac{n(B)}{n(S)} = \frac{4}{120} = 0.033$$

iii) Let C be the event of drawing one green ball

$$n(C) = \binom{4}{2}\binom{6}{1} = 36$$

$$P(C) = \frac{n(C)}{n(S)} = \frac{36}{120} = 0.30$$

iv) Let D be the event of drawing no red ball

$$n(D) = \binom{4}{0}\binom{6}{3} = 20$$

$$\therefore P(D)=\frac{n(D)}{n(S)}=\frac{20}{120}=0.17$$

Example 6.7: If two fair dice are thrown, what is the probability of getting

i) a double six. ii) a sum of 8 or more dots.

Solution: Sample space is given by

$$S = \{(1,1),(1,2),(1,3),(1,4),(1,5),(1,6),\\ (2,1),(2,2),(2,3),(2,4),(2,5),(2,6),\\ (3,1),(3,2),(3,3),(3,4),(3,5),(3,6),\\ (4,1),(4,2),(4,3),(4,4),(4,5),(4,6),\\ (5,1),(5,2),(5,3),(5,4),(5,5),(5,6),\\ (6,1),(6,2),(6,3),(6,4),(6,5),(6,6)\}$$

$$\Rightarrow \quad n(S) = 36$$

i) Let A be the event that a double six occurs

$$A = \{(6,6)\} \Rightarrow n(A) = 1$$

$$P(A)=\frac{n(A)}{n(S)}=\frac{1}{36}$$

ii) Let B be the event that a sum 8 or more dots occurs.

$$B = \{(2,6),(3,5),(4,4),(5,3),(6,2),(3,6)(4,5)(4,6)\\ (5,4),(5,5),(5,6),(6,3),(6,4),(6,5),(6,6)\}$$

$$\Rightarrow n(B) = 15$$

$$\therefore \; P(B)=\frac{n(B)}{n(S)}=\frac{15}{36}=\frac{5}{12}$$

Example 6.8: Six white balls and four black balls which are indistinguishable apart from colour, are placed in a bag. If six balls are taken from the bag, find the probability that their being three white and three black.

Solution:

White	Black	Total
6	4	10

$$\Rightarrow n(S)=\binom{10}{6}=210$$

Let A be the event that three white and three black balls are taken

$$n(A) = \binom{6}{3}\binom{4}{3} = 80$$

$$P(A) = \frac{n(A)}{n(S)} = \frac{80}{210} = \frac{8}{21}$$

Example 6.9: A fair die is tossed. Find the probability that the number on the uppermost face is not six.

Solution: Sample space is given by

$$S = \{1,2,3,4,5,6\} \Rightarrow n(S) = 6$$

Let A be the event that the uppermost face is 6 and $\overline{A}$ be the event that face is not 6. Then

$$A = \{6\} \Rightarrow n(A) = 1$$

$$P(A) = \frac{n(A)}{n(S)} = \frac{1}{6} \text{ and } P(\overline{A}) = 1 - P(A) = 1 - \frac{1}{6} = \frac{5}{6}$$

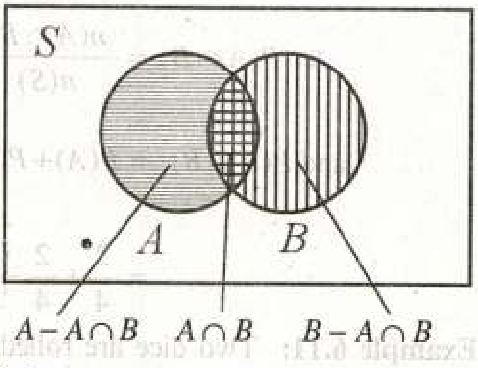
Theorem not Mutually Exclusive Events

Statement: If A and B are two not mutually exclusive events, the probability atleast one of them happens is the probability that event A occurs plus the probability that event B occurs minus the probability that both A and B occur simultaneously i.e.,

$$P(A \cup B) = P(A) + P(B) - P(A \cap B)$$

Proof: The event A or B can be expressed as the union of two mutually exclusive events A and $B - A \cap B$, then

$$A \cup B = A \cup (B - A \cap B)$$



By taking probability on both the sides we have,

$$P(A \cup B) = P[A \cup (B - A \cap B)]$$

$$= P(A) + P(B - A \cap B) \quad \text{........ (i)}$$

We can express B as a union of two mutually exclusive events $A \cap B$ and $B - (A \cap B)$ then

$$B = (A \cap B) \cup (B - A \cap B)$$

By taking probability on both the sides we have,

$$P(B) = P[(A\cap B)\cup(B-A\cap B)]$$

$$P(B) = P(A\cap B)+P(B-(A\cap B))$$

$$P(B) = P[A\cap B)+P(B-A\cap B)]$$

$$P(B) - P(A\cap B)=P(B-A\cap B)$$

By putting the value of $P(B-A\cap B)$ in equation (i) we get,

$$P(A\cup B)=P(A)+P(B)-P(A\cap B)$$

Example 6.10: A coin is tossed twice, points of the sample space are HH, HT, TH, TT and each sample point with probability $\frac{1}{4}$.

If A and B are the events that head at first coin and tail on second coin respectively. Then find $P(A\cup B)$.

Solution: Sample space is given by

$$S = \{\text{HH},\text{HT},\text{TH},\text{TT}\}\Rightarrow n(S) = 4$$

$$\therefore\ A=\{\text{HH, HT}\} \qquad \Rightarrow n(A)=2$$

$$\therefore\ P(A)=\frac{n(A)}{n(S)}=\frac{2}{4}$$

$$B=\{\text{TT, HT}\}\Rightarrow n(B)=2$$

$$\therefore\ P(B)=\frac{n(B)}{n(S)}=\frac{2}{4}$$

$$A\cap B=\{\text{HT}\}\Rightarrow n(A\cap B)=1$$

$$\therefore\ P(A\cap B)=\frac{n(A\cap B)}{n(S)}=\frac{1}{4}$$

and $P(A\cup B) = P(A)+P(B)-P(A\cap B)$

$$=\frac{2}{4}+\frac{2}{4}-\frac{1}{4}=\frac{3}{4}$$

Example 6.11: Two dice are rolled. If A and B are respectively the events that the sum of points is 8 and both dice should give odd numbers, Then find $P(A\cup B)$.

Solution: Sample space is shown in the example 6.7, where we see

$$n(S) = 36$$

$$A = \{(2,6),(3,5),(4,4),(5,3),(6,2)\} \Rightarrow n(A) = 5$$

$$\therefore \quad P(A) = \frac{n(A)}{n(S)} = \frac{5}{36}$$

$$\therefore B = \{(1,1),(1,3),(1,5),(3,1),(3,3),(3,5),(5,1),(5,3),(5,5)\} \Rightarrow n(B) = 9$$

$$\therefore P(B) = \frac{n(B)}{n(S)} = \frac{9}{36}$$

$$A \cap B = \{(3,5),(5,3)\} \Rightarrow n(A \cap B) = 2$$

$$\therefore P(A \cap B) = \frac{n(A \cap B)}{n(S)} = \frac{2}{36}$$

$$P(A \cup B) = P(A) + P(B) - P(A \cap B)$$

$$= \frac{5}{36} + \frac{9}{36} - \frac{2}{36} = \frac{1}{3}$$

Addition Theorem for Mutually Exclusive Events

If A and B are two mutually exclusive events then the probability either of them happening is the sum of their respective probabilities i.e.,

$$\boxed{P(A \cup B) = P(A) + P(B)}$$

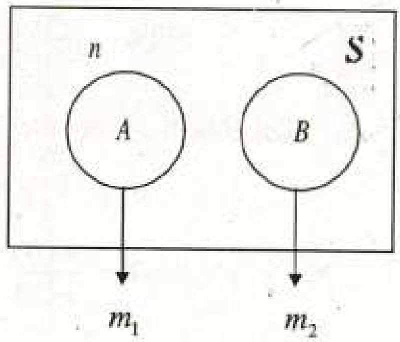
Proof: Let n be the number of sample points in a sample space S. Let A and B be the two mutually exclusive events in the sample S, such that event A contains m_1 sample points and event B contains m_2 sample points. Then $A \cup B$ will contain the sample points belonging to either A or B.

As A and B are two mutually exclusive events. i.e., $A \cap B = \Phi$

The sample points for $A \cup B$ will be $m_1 + m_2$

$$P(A \cup B) = \frac{m_1 + m_2}{n} = \frac{m_1}{n} + \frac{m_2}{n} = P(A) + P(B)$$

$$\therefore P(A \cup B) = P(A) + P(B)$$



Example 6.12: A pair of dice are rolled. Find the probability that the sum of the uppermost dots is either 6 or 9.

Solution: Sample space is shown in the example 6.7, where we see

$$n(S) = 36$$

Let A be the event that the sum of the uppermost dots is 6, then

$$A = \{(1,5),(2,4),(3,3),(4,2),(5,1)\} \Rightarrow n(A) = 5$$

$$\therefore\ P(A) = \frac{n(A)}{n(S)} = \frac{5}{36}$$

Let B be the event that the sum of the uppermost dots is 9, then

$$B = \{(3,6),(4,5),(5,4),(6,3)\} \Rightarrow n(B) = 4$$

$$\therefore\ P(B) = \frac{n(B)}{n(S)} = \frac{4}{36} = \frac{1}{9}$$

Since A and B are mutually exclusive events.

$$\therefore P(A \cup B) = P(A) + P(B) = \frac{5}{36} + \frac{4}{36} = \frac{1}{4}$$

Example 6.13: A pair of dice is thrown. Find the probability of getting a total of either 5 or 11.

Solution: Sample space is shown in the example 6.7, where we see

$$n(S) = 36$$

Let A be the event that a total of 5 occurs, then

$$A = \{(1,4),(2,3),(3,2),(4,1)\} \Rightarrow n(A) = 4$$

$$\therefore\ P(A) = \frac{n(A)}{n(S)} = \frac{4}{36} = \frac{1}{9}$$

Let B be the event that a total of 11 occurs.

$$B = \{(5,6),(6,5)\} \Rightarrow n(B) = 2$$

$$\therefore\ P(B) = \frac{n(B)}{n(S)} = \frac{2}{36} = \frac{1}{18}$$

As events A and B are mutually exclusive, then

$$P(A \cup B) = P(A) + P(B) = \frac{1}{9} + \frac{1}{18} = \frac{1}{6}$$

Example 6.14: Three horses A, B and C are in a race. A is twice as likely to win as B and B is twice as likely to win as C, then

i) What are their respective chance of winning.

ii) What is the probability that B or C wins

Solution: Let $P(C) = P$

$$P(B) = 2P(C) = 2P$$
$$P(A) = 2P(B) = 2(2P) = 4P$$

i) Since A, B and C are mutually exclusive and collectively exhaustive events. Therefore, the probabilities must be equal to one i.e.,

$$P(A) + P(B) + P(C) = 1$$
$$\Rightarrow 4P + 2P + P = 1$$
$$\Rightarrow 7P = 1$$
$$\Rightarrow P = \frac{1}{7}$$
$$\therefore P(A) = \frac{4}{7},\ P(B) = \frac{2}{7} \text{ and } P(C) = \frac{1}{7}$$

ii) As B and C are mutually exclusive events, so

$$P(B \cup C) = P(B) + P(C) = \frac{2}{7} + \frac{1}{7} = \frac{3}{7}$$

6.4.1 Conditional Probability

If two events A and B are defined on a sample space S and if probability of B is not equal to zero, then the conditional probability of an event A given that B has occurred is written as $P(A/B)$ and is defined as:

$$P(A/B) = \frac{P(A \cap B)}{P(B)}, \text{ where } P(B) > 0$$

If P(B)=0, the conditional probability is undefined

Multiplication Theorem For Independent Events

If A and B are two independent events, then the probability that A and B happen is the product of their respective probabilities, i.e.,

$$\boxed{P(A\cap B) = P(A)P(B)}$$

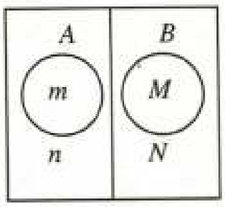
Proof: Let event A has n possible outcomes and m favourable outcomes and event B has N possible outcomes and M favourable outcomes then,

$$P(A) = \frac{m}{n} \text{ and } P(B) = \frac{M}{N}$$

As A and B are independent events so there will be nN possible outcomes for the joint events A and B. As each of n possible outcomes for A are associated with N possible outcomes for B. Similarly, each favourable outcomes for A is associated with each favourable outcomes for B. To have the favourable outcomes for compound event $A\cap B$. Then the total favourable outcomes for A and B are mM.

$$P(A\cap B) = \frac{mM}{nN} = \frac{m}{n}\times\frac{M}{N}$$

$$\text{or } P(A\cap B) = P(A)\,P(B)$$



Multiplication Theorem for not Independent Events

If A and B are two not mutually independent events, then the probability that both A and B happens is the probability of event A multiplied by the conditional probability of B given that event A has already occurred or is the probability of event B multiplied by the conditional probability of event A given that event B has already occurred i.e.,

$$\boxed{P(A\cap B) = P(A)P(B/A) \text{ or } P(A\cap B) = P(B)P(A/B)}$$

Proof: Let us have n sample points in a sample space S and A and B are not independent events such that event A has m_1, event B has m_2 and $A\cap B$ has m_3 sample points.

Then, $P(A)=\frac{m_1}{n}$ and $P(B)=\frac{m_2}{n}$

$$P(A\cap B)=\frac{m_3}{n}$$

Now, $P(A\cap B)=\frac{m_3}{n}\times\frac{m_1}{m_1}=\frac{m_3}{m_1}\times\frac{m_1}{n}$

Here, $P(A)=\frac{m_1}{n}$ and $\frac{m_3}{m_1}=P(B/A)$

$\therefore\ P(A\cap B)=P(B/A)\ P(A)$

or $P(A\cap B)=\frac{m_3}{n}=\frac{m_3}{n}\times\frac{m_2}{m_2}=\frac{m_3}{m_2}\times\frac{m_2}{n}$

Here, $\frac{m_2}{n}=P(B)$ and $\frac{m_3}{m_2}=P(A/B)$

$\therefore\ P(A\cap B)=P(A/B)\ P(B)$

Example 6.15: Two *a*'s and two *b*'s are arranged in order, all arrangements are equally likely given that the last letters in order is *b*. Find the probability that 2 *a*'s are together.

Solution: $S=\{aabb, abba, bbaa, baab, baba, abab\}\Rightarrow n(S)=6$

Let *A* be the event that the two *a*'s are together

$$A=\{aabb, bbaa, baab\}\qquad\Rightarrow n(A)=3$$

$$P(A)=\frac{n(A)}{n(S)}=\frac{3}{6}=\frac{1}{2}$$

Let *B* be the event that the last letter is b

$$B=\{aabb, abab, baab\}\qquad\Rightarrow n(B)=3$$

$$\therefore\quad P(B)=\frac{n(B)}{n(S)}=\frac{3}{6}$$

$$A\cap B=\{aabb, baab\}\qquad\Rightarrow n(A\cap B)=2$$

$$\therefore\ P(A\cap B)=\frac{n(A\cap B)}{n(S)}=\frac{2}{6}=\frac{1}{3}$$

$$P(A/B)=\frac{P(A\cap B)}{P(B)}=\frac{1/3}{1/2}=\frac{2}{3}$$

Example 6.16 In tossing two coins find:

i) The probability of two heads given that a head on the first coin.

ii) The probability of two heads given that atleast one head.

Solution: Sample space is

$$S = \{HH, HT, TH, TT\} \Rightarrow n(S) = 4$$

Let A be the event that the head appears on the first coin.

$$A = \{HH, HT\} \Rightarrow n(A) = 2$$

$$\therefore P(A) = \frac{n(A)}{n(S)} = \frac{2}{4} = \frac{1}{2}$$

Let B be the event that two heads appear

$$B = \{HH\} \Rightarrow n(B) = 1$$

$$\therefore P(B) = \frac{n(B)}{n(S)} = \frac{1}{4}$$

$$A \cap B = \{HH\} \Rightarrow n(A \cap B) = 1$$

$$\therefore P(A \cap B) = \frac{n(A \cap B)}{n(S)} = \frac{1}{4}$$

$$P\left(B/A\right) = \frac{P(A \cap B)}{P(A)} = \frac{1}{2}$$

ii) Let C be the event that atleast one head appears

$$C = \{HH, HT, TH\} \Rightarrow n(C) = 3$$

$$P(C) = \frac{n(C)}{n(S)} = \frac{3}{4}$$

$$B = \{HH\} \Rightarrow n(B) = 1$$

$$P(B) = \frac{n(B)}{n(S)} = \frac{1}{4}$$

$$\therefore P(B/A) = \frac{P(A \cap B)}{P(A)} = \frac{1/4}{1/2} = \frac{1}{2}$$

$$B \cap C = \{HH\} \Rightarrow n(B \cap C) = 1$$

$$P(B \cap C) = \frac{n(B \cap C)}{n(S)} = \frac{1}{4}$$

$$\therefore P(B/C) = \frac{P(B \cap C)}{P(C)} = \frac{1/4}{3/4} = \frac{1}{3}$$

Example 6.17: A and B are two independent events. If $P(A) = 0.40$, $P(B) = 0.30$. Find the probabilities

i) $P(A \cap B)$ *ii)* $P(A/B)$ *iii)* $P(B/A)$

iv) $P(A \cup B)$ *v)* $P(\overline{A} \cap \overline{B})$ *iv)* $P(\overline{A}/\overline{B})$

Solution: We have

$$P(A) = 0.40, \quad P(B) = 0.30$$

Since A and B are independent events, therefore,

i) $P(A \cap B) = P(A)\,P(B) = (0.40)\,(0.30) = 0.12$

ii) $P(A/B) = \dfrac{P(A \cap B)}{P(B)} = \dfrac{0.12}{0.30} = 0.40$

iii) $P(B/A) = \dfrac{P(A \cap B)}{P(A)} = \dfrac{0.12}{0.40} = 0.3$

iv) $P(A \cup B) = P(A) + P(B) - P(A \cap B)$

$= 0.40 + 0.30 - 0.12 = 0.70 - 0.12$

$= 0.58$

v) $P(\overline{A} \cap \overline{B}) = P(\overline{A \cup B})$

$= 1 - P(A \cup B) = 1 - 0.58 = 0.42$

vi) $P(\overline{A}/\overline{B}) = \dfrac{P(\overline{A} \cap \overline{B})}{P(\overline{B})} = \dfrac{P(\overline{A} \cap \overline{B})}{1 - P(B)}$

$P(\overline{A}/\overline{B}) = \dfrac{0.42}{1 - 0.30} = \dfrac{0.42}{0.70} = 0.6$

Example 6.18: Three missiles are fired at a target. If the probabilities of hitting the target are 0.4, 0.5 and 0.6 respectively and if the missiles are fired independently what is the probability that at least two missiles hit the target?

Solution:

$$P(A) = 0.40 \Rightarrow P(\overline{A}) = 0.6$$
$$P(B) = 0.5 \Rightarrow P(\overline{B}) = 0.5$$
$$P(C) = 0.6 \Rightarrow P(\overline{C}) = 0.4$$

Let D be the event that at least two shots hit the targets.

$$P(D) = P(A \cap B \cap \overline{C}) + P(A \cap \overline{B} \cap C) + P(\overline{A} \cap B \cap C) + P(A \cap B \cap C)$$
$$= P(A)P(B)P(\overline{C}) + P(A)P(\overline{B})P(C) + P(\overline{A})P(B)P(C) + P(A)P(B)P(C)$$
$$= 0.4\times0.5\times0.4 + 0.4\times0.5\times0.6 + 0.6\times0.5\times0.6 + 0.4\times0.5\times0.6$$
$$= 0.08+0.12 + 0.18 + 0.12$$

or $P(D) = 0.50$

Example 6.19: A purse contains 2 silver,4 copper and a second purse contains 4 silver, 3 copper coins. If a coin is selected at random from one of the purses. What is the probability that it is a

i) Silver coin ii) Copper coin.

Solution:

	Purse I	**Purse II**
Silver coin	2	4
Copper coin	4	3
Total coin	6	7
P (Purse I)	$1/2$	$1/2$

i) Let A be the event that the selected coin is silver

$$P(A) = P(\text{Purse I})\,P\left(\text{Silver coin}/_{\text{Purse I}}\right) + P(\text{Purse II})\,P\left(\text{Silver coin}/_{\text{Purse II}}\right)$$
$$= \left(\frac{1}{2}\times\frac{2}{6}\right)+\left(\frac{1}{2}\times\frac{4}{7}\right)=\frac{2}{12}+\frac{4}{14}=\frac{38}{84}=\frac{19}{42}$$

ii) Let B be the event that the coin selected is a copper, then

$$P(B) = P(\text{Purse I})\, P\left(\text{Coper coin}/_{\text{Purse I}}\right) + P(\text{Purse II})\, P\left(\text{Copper coin}/_{\text{Purse II}}\right)$$

$$= \left(\frac{1}{2}\times\frac{4}{6}\right)+\left(\frac{1}{2}\times\frac{3}{7}\right)=\frac{4}{12}+\frac{3}{14}=\frac{23}{42}$$

Exercise 6

Ans on Page 255

6.1. i) Define permutation and combination and discriminate between these two?

ii) Let $A = \{1, 3, 5, 7\}$, $B = \{2, 4, 6, 8\}$, $C = \{1,2,3,4,5\}$ and $S = \{1, 2, 3, 4, 5, 6, 7, 8\}$

list the elements of the following:-

a) $A \cap B$ b) $C \cup A$ c) $\overline{A \cap C}$ d) $B \cap C$

6.2 Evaluate the following

i) 7! ii) 16!/8! iii) 17!/8!.4!

iv) ${}^{9}P_{6}$ v) ${}^{15}P_{6}$ vi) ${}^{5}P_{5}$

vii) ${}^{5}c_{3}$ viii) ${}^{52}c_{13}$ ix) ${}^{11}c_{4}$

6.3 Let $A = \{1, 4\}$, $B = \{2, 3\}$, $C = \{3\}$ be the subsets of the universal set $S = \{1, 2, 3, 4\}$, find

i) $A \times B$ ii) $B \times A$ iii) $A \times A$ iv) $B \times B$

6.4 What do you understand by

i) sample space ii) event iii) sample point iv) simple and compound event?

6.5 Define mutually exclusive, independent and dependent events.

6.6 Explain with examples the terms; random experiment, sample space and an event.

6.7 State and prove the multiplicative law of probability for two events A and B that are not statistically independent.

6.8 Find the probability of each of the following:

i) A head appears in tossing a fair coin.

ii) A 5 appears in rolling a six faced cubical dice.

iii) An even number appears when a perfect cubical die is rolled.

6.9 What is the probability of selecting a card of diamonds from a pack of playing cards consisting of the usual 52 cards.

6.10 Show that in a single throw with two die, the chance of throwing more than 7 is equal to that of throwing less than 7.

6.11 A bag contains 12 balls of which 3 are marked, if 5 balls are drawn out together. What is the probability that 3 of the marked balls are among these 5?

6.12 What is the probability of throwing either 7 or more than 10 with two dice?

6.13 A bag containing 2 red, 3 green, 5 blue and 2 yellow balls. Find the probability that balls of all colours, are represented in a sample if four balls are selected at random.

6.14 A bag contains 5 white and 7 black balls. If 3 balls are drawn from the bag, what is the probability that;

i) All are white. ii) Two are white and one is black.

iii) All are of the same colour?

6.15 Determine the probability for the following events:

i) the sum 8 appears in a single toss of a pair of fair dice.

ii) A sum 7 or 11 comes up in a single toss of a pair of fair dice.

iii) A ball drawn at random from a bag containing 5 red, 6 white, 4 blue and 3 orange balls is either red or blue.

6.16 Two cards are drawn at random from a well shuffled pack of 52 cards. Find the probability that:

i) One is king and other is queen. ii) Both are of same colour.

iii) Both are of different colours.

6.17 A bag contains 9 white and 12 black balls. Find the probability of drawing 5 black balls out of the bag containing 21 balls.

6.18 From a bag containing 5 white, and 3 black balls, 2 are drawn at random. Find the chance that both are of the same colour.

6.19 A set of eight cards contains one joker. *A* and *B*, are two players, choose 5 cards at random, *B* takes the remaining 3 cards, what is the probability that *A* has a joker?

6.20 From a pack of 52 cards, two cards are drawn. What is the probability that one is king and the other is queen.

6.21 In a poker hand consisting of 5 cards. What is the probability of holding?

i) 2 aces and 2 kings.

ii) 5 spades.

6.22 A bag containing 14 identical balls, out of which 4 are red, 5 black and 5 white balls, if three balls are drawn from the bag. Find the chance that

i) 3 are red. ii) At least two are white.

6.23 A marble is drawn at random from a box containing 10 red, 30 white, 20 blue and 15 orange marbles. Find the probability that it is:

i) orange or red. ii) not red or orange.

ii) not blue. iv) red, white or blue.

6.24 What is meant by conditional probability?

6.25 State and prove multiplication laws of probabilities for independent and dependent events.

6.26 *A* and *B* can solve 60% and 80% of the problems in a book respectively. What is the probability that either *A* or *B* can solve a problem chosen at random?

6.27 A class contains 10 men and 20 women out of which half men and half women have brown eyes. Find the probability that a person chosen at random is a man or has brown eyes.

6.28 A box contains 9 tickets numbered 1 to 9. If 3 tickets are drawn from the box one at a time, find the probability that they are alternately either odd even odd or even odd even.

6.29 Two drawings each of 3 balls are made from a bag containing 5 white and 8 black balls. The balls are not being replaced before the next trial. What is the probability that the first drawing will give 3 white balls and the second drawing will give three black balls.

6.30 Three balls are drawn successively from a box containing 6 red balls, 4 white balls and 5 blue balls. Find the probability that they are drawn in the order red, white and blue if each ball is

i) replaced ii) not replaced.

6.31 One bag contains 4 white and 2 black balls another bag contains 3 white and 5 black balls. If one ball is drawn from each bag, find the probability that.

i) both are white ii) both are black.

6.32 Two urns contain respectively 3 white and 7 red balls 15 black and 10 white balls, 6 red and 9 black balls. One ball is taken from each urn. What is the probability that both will be of the same colour?

6.33 The probability that a man will alive in 25 years is 3/5 and that his wife will be alive in 25 years is 2/3, find the probability that.

i) Both will alive in 25 years.

ii) Only the man will alive in 25 years.

iii) Only the wife will alive in 25 years.

iv) Atleast one will alive in 25 years.

v) None of them will alive in 25 years.

vi) At the most one will alive in 25 years.

6.34 Three cards are drawn at random from an ordinary pack of 52 cards. Find the probability that they will consists of a Jack, a queen and a king.

6.35 Two cards are drawn from a well shuffled pack of 52 cards. Find the probability that they are both aces if the first cards drawn is:

i) replaced. ii) not replaced.

6.36 Urn A contains 5 red balls and 3 white balls and urn B contains 2 red balls and 6 white balls.

i) If a ball is drawn from each urn what is the probability that both are of the same colour?

ii) If two balls are drawn from each urn what is the probability that all 4 balls are of the same colour?

6.37 Three Ghori missiles are fired at a target. If the probabilities of hitting the target are 0.4, 0.5 and 0.6 respectively and the missiles are fired independently, what is the probability that atleast 2 missiles hit the target?

6.38 Assume that X is a number chosen at random from the set of integers between 1 and 14 respectively. What is the probability that

i) X is a single digit number. ii) X is a multiple of 5 or 6.

6.39 What are the odds for the occurrence of an even if its probability is 4/7?

6.40 Suppose that it is 9 to 7 against a person A who is now 35 years of age living till he is 65 and 3 to 2 against a person B now 45 years of age living till he is 75. Find the probability that one of these persons will be alive 30 years hence.

6.41 A purse contains 2 silver and 4 copper and second purse contains 4 silver and 3 copper coins. If a coin is selected at random from one of the purses. What is the probability that it is a

i) Silver coin. ii) Copper coin.

6.42 One purse contains 1 sovereign and 3 shillings, a second purse contains 2 sovereign and 4 shillings and third purse contains 3 sovereign and 1 shilling. If the coin taken out of the purse is selected at random. Find the chance that it is sovereign.

6.43 The probability that a student will get a grade of A, B or C in statistics course are 0.09, 0.15 and 0.53 respectively, what is the probability that the student will get a grade lower than C.

6.44 3 coins are tossed, what is the probability of getting atleast one head?

6.45 A and B play 12 games of chess out of which 6 are won by A and 4 by B and two games end in a tie. They agree to play a tournament consisting of 3 games. Find the probability that

i) A wins all the 3 games. ii) Two games end in tie.

iii) A and B won alternately. iv) B wins atleast 1 game.

6.46 For two independent events A and B, $P(A) = 0.25$ and $P(B) = 0.40$. Find $P(A \cap B)$?

6.47 For 2 rolls of a balanced die, find the probability of getting 1st a five and then a number less than 4.

6.48 If two cards are drawn from an ordinary deck of 52 cards. What is the probability that both will be diamonds, if the drawing is without replacement?

6.49 A, B and C take turns in throwing a die for a prize to be given to one who first obtains 6. Compare their chances of success.

6.50 First bag contains 4 white balls and 3 black balls and second bag contains 3 white and 5 black balls. One ball is drawn from the first bag and placed unseen in the second bag. What is the probability that a ball now drawn from the second bag is black?

6.51 From a bag containing 4 white and 5 black balls, 2 balls are drawn at random. Find the probability that they are of same colour.

6.52 3 groups of children contains respectively 3 girls and 1 boy, 2 girls and 2 boys and 1 girl and 3 boys. One child is selected at random from each group. Find the chance that the selected group comprises of one girl and two boys.

6.53 A can hit a target 4 times in 5 shots, B can hit 2 times in 5 shots and C can hit 2 times in 4 shots. Find the probability that

i) 2 shots hit. ii) at least two shots hit.

6.54 Fill in the blanks:

i) A set is any _________ collection of _________ things.

ii) A set containing only one element is called _________ set.

iii) A set of subsets of set is called _________ of the set.

iv) A _________ event contains more than one element.

v) Two events are _________ if they have no common point.

vi) If $A \cup B = S$ then A and B are _________ events.

vii) If $n(A) = n(B)$, then A and B are _________ events.

viii) The number of subsets of a set containing n points are _________.

ix) The orderly arrangements of r distinct things out of n are called _________ and denoted by _________.

x) A non-orderly arrangement of things is called _________.

6.55 Against each statement, write T for true and F for false statement.

i) The null set is also named as the impossible event.

ii) A part of a set is called improper subset of the set.

iii) A subset of sample space is called sure event.

iv) An event consisting of only one sample point is called compound event.

v) If a coin is tossed four times, the number of sample points will be 22.

vi) If A and B are mutually exclusive events then, $P(A \cup B) = P(A) + P(B)$.

vii) The probability of drawing a red card from 52 cards is 13/15.

viii) The complementary events are always mutually exclusive events.

ix) When a coin is tossed, the sample space is {HH, HT}.

x) If A and B are independent, then $P(A \cup B) = P(A) + P(B)$.